

Exercise walk-through

16.1

Improper integrals and geometric applications

GRE · GRE Subject Test · GRE Mathematics

6. Choose and justify a method

Choose and justify a method

An infinite integration endpoint is replaced by a finite bound and a limit. At a singular point inside the interval, split the integral and require both one-sided integrals to converge separately. Symmetric cancellation can define a Cauchy principal value but does not prove convergence of the ordinary improper integral. For $1/x$ across zero, the two sides diverge even though symmetric cutoffs cancel.

9. Choose and justify a method

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For rotation around the x-axis, disks or washers integrate $\pi(R^2 - r^2) dx$. Cylindrical shells use 2π times radius times height and integrate in the matching variable. Select the method by the geometry and verify nonnegative radii. Area between curves integrates upper minus lower, splitting where their order changes. A signed integral is not automatically geometric area.

15. Reasoning and check

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Split at 1. Near zero, convergence requires $p < 1$; near infinity, it requires $p > 1$. No real p meets both, and $p=1$ diverges logarithmically at both ends. Both one-sided parts must converge independently. For $1/x$ across zero, the negative and positive parts diverge separately, so the ordinary improper integral diverges. Symmetric cutoffs cancel to zero, yielding a principal value only. The zero is not the value of the ordinary integral.

Exercise 2.1

2.1 For which p does the integral of x^{-p} from zero to 1 converge?

[1]

- **A** $p > 1$
- **B** $p \geq 1$
- **C** $p < 1$
- **D** $p \leq 1$
- **E** All real p

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2.1 C. The lower-bound power limit is finite when $1-p > 0$; $p=1$ gives logarithmic divergence. Conditions $p > 1$ and $p \geq 1$ belong to a different endpoint or include logarithmic divergence. $p \leq 1$ wrongly includes $p=1$; all real p includes divergent examples.

Exercises 2.2·2.3

2.2 Rotating the region under $y=x^2$ from $x=0$ to $x=1$ around the x -axis gives which volume? [1]

- A $\pi/2$
- B $\pi/3$
- C $\pi/4$
- D $\pi/5$
- E $2\pi/3$

2.3 For which real p does $\int_0^\infty x^{-p} dx$ converge? Compare the ordinary improper integral

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2.2 D. Disk radius is x^2 , so $V=\pi \int_0^1 x^4 dx=\pi/5$. Disk radius is x squared and

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3.1 Disks give $V = \pi \lim_{b \rightarrow \infty} \int_1^b x^{-2} dx = \pi \lim(1 - 1/b) = \pi$. For surface area, $y' = -1/x^2$ and

$$S = 2\pi \int_1^\infty x^{-1} \sqrt{1 + x^{-4}} dx.$$

Its nonnegative integrand is at least $2\pi/x$, whose integral diverges. Comparison proves infinite lateral area. This includes neither an end cap nor a claim that finite volume implies finite area.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.