

Exercise walk-through

15.1

Taylor expansions and power-series endpoints

GRE · GRE Subject Test · GRE Mathematics

6. Choose and justify a method

Choose and justify a method

The Taylor polynomial of degree m at a is the sum of $f^{(k)}(a)(x-a)^k/k!$ for k from zero to m . The factorial belongs to each coefficient. If the next derivative is bounded in magnitude by M between a and x , the Lagrange remainder has magnitude at most $M|x-a|^{m+1}/(m+1)!$. Smoothness alone does not guarantee that the infinite Taylor series equals the function everywhere.

8. Choose and justify a method

Choose and justify a method

Within the open interval of convergence, termwise differentiation and integration preserve the radius. They can change whether endpoints are included. Start from the geometric series $1/(1-x) = \sum x^n$ for $|x| < 1$, then integrate from zero to x to obtain $-\ln(1-x) = \sum x^n/n$ for $n \geq 1$. Check the integration constant and the real logarithm domain.

14. Reasoning and check

Reasoning and check

The ratio test gives radius one. At 1 the original harmonic series diverges; at -1 the alternating series converges. Thus the original interval is $[-1, 1)$. The derivative is $\sum_{n=1}^{\infty} x^{n-1} = 1/(1-x)$ only for $|x| < 1$; its terms do not tend to zero at either endpoint. $P_2(0.1) = 1 + 0.1 + 0.1^2/2 = 1.105$. On $[0, 0.1]$, the third derivative is $e^x < 2$. The Lagrange bound gives $|R_2| \leq 2(0.1)^3/6 = 1/3000$. The bound is deliberately conservative, not a claim of exact error.

Exercise 2.1

2.1 What is the coefficient of x^3 in the Maclaurin series of $e^{(2x)}$? [1]

- **A** $1/6$
- **B** $1/3$
- **C** $2/3$
- **D** $4/3$
- **E** 8

Exercise 2.1

2.1 What is the coefficient of x^3 in the Maclaurin series of $e^{(2x)}$? [1]

- A $1/6$
- B $1/3$
- C $2/3$
- D $4/3$
- E 8

2.1 D. The coefficient is $2^3/3!=8/6=4/3$; the factorial cannot be omitted. One-sixth is the coefficient for e^x ; one-third and two-thirds omit some powers of two; eight omits division by factorial three.

Exercises 2.2·2.3

2.2 Which is the interval of convergence of $\sum_{n \geq 1} x^n/n^2$?

[1]

- **A** $(-1,1)$
- **B** $[-1,1)$
- **C** $(-1,1]$
- **D** $[-1,1]$
- **E** $\mathbb{R}$

2.3 Find the convergence interval of $\sum_{n=1}^{\infty} x^n/n$ and of its termwise derivative. Approx-

Exercises 2.2·2.3

2.2 Which is the interval of convergence of $\sum x^n/n^2$ for $n \geq 1$?

[1]

- **A** $(-1,1)$
- **B** $[-1,1)$
- **C** $(-1,1]$
- **D** $[-1,1]$
- **E** $\mathbb{R}$

2.2 D. The radius is 1 and both endpoints converge absolutely by comparison with $\sum 1/n^2$. Each open or half-open interval wrongly discards at least one absolutely convergent endpoint; all of $\mathbb{R}$ includes inputs with magnitude greater than one where terms fail to tend to zero.

2.3 Find the convergence interval of $\sum_{n=1}^{\infty} x^n/n$ and of its termwise derivative. Approx-

2.3 The ratio test gives radius one. At 1 the original harmonic series diverges; at -1 the alternating series converges. Thus the original interval is $[-1, 1)$. The derivative is $\sum_{n=1}^{\infty} x^{n-1} = 1/(1-x)$ only for $|x| < 1$; its terms do not tend to zero at either endpoint. $P_2(0.1) = 1 + 0.1 + 0.1^2/2 = 1.105$. On $[0, 0.1]$, the third derivative is $e^x < 2$. The Lagrange bound gives $|R_2| \leq 2(0.1)^3/6 = 1/3000$. The bound is deliberately conservative, not a claim of exact error.

Exercise 3.1

3.1 Evaluate $\lim_{x \rightarrow 0} (\cos x - 1 + x^2/2)/x^4$. Explain why replacing cosine by only $1 - x^2/2$ cannot determine this limit. [4]

Exercise 3.1

3.1 Evaluate $\lim_{x \rightarrow 0} (\cos x - 1 + x^2/2)/x^4$. Explain why replacing cosine by only $1 - x^2/2$ cannot determine this limit. [4]

3.1 Taylor's theorem near zero gives $\cos x = 1 - x^2/2 + x^4/24 + O(x^6)$. Subtraction cancels the constant and quadratic terms, leaving $x^4/24 + O(x^6)$. After division the limit is $1/24$. A truncation with remainder merely $o(x^2)$ gives no control after division by x^4 ; retain and bound the first surviving order.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.