

Exercise walk-through

14.1

Open sets, compactness and connectedness

GRE · GRE Subject Test · GRE Mathematics

8. Choose and justify a method

Choose and justify a method

Compactness means every open cover has a finite subcover. In Euclidean $\mathbb{R}^n$, Heine–Borel makes this equivalent to closed and bounded. In a metric space, compactness is equivalent to sequential compactness; completeness and boundedness alone do not suffice in arbitrary metric spaces. A continuous image of a compact set is compact, giving attained maxima and minima for real continuous functions on a nonempty compact domain.

9. Choose and justify a method

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Connected sets cannot be separated into two disjoint nonempty relatively open parts; connected subsets of $\mathbb{R}$ are precisely intervals. A continuous image of a connected set is connected, which yields the intermediate value theorem. Path connectedness implies connectedness, but not conversely in every space. A compact set need not be connected, and a connected set need not be compact; a finite two-point set and an open interval supply the contrasting cases.

16. Reasoning and check

Reasoning and check

The finite set $\{0, 2\}$ is compact but disconnected. The interval $(0, 1)$ is connected but not compact. The continuous identity function on $(0, 1)$ attains neither a maximum nor a minimum. Connectedness makes a continuous real image an interval and therefore supplies intermediate values. Compactness instead makes a nonempty continuous real image attain its extreme values. Neither property substitutes for the other.

Exercise 2.1

2.1 What is the boundary of $(0,1)$ as a subset of $\mathbb{R}$?

[1]

- **A** The empty set
- **B** $\{0\}$
- **C** $\{1\}$
- **D** $\{0,1\}$
- **E** $[0,1]$

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2.1 D. The closure is $[0,1]$ and the interior is $(0,1)$; removing the latter leaves both endpoints. Empty omits both endpoints; the singleton options omit one endpoint; $[0,1]$ incorrectly includes interior points.

Exercises 2.2·2.3

2.2 Which subset of $\mathbb{R}$ is connected but not compact?

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- **A** $\{0,1\}$
- **B** $[0,1]$
- **C** $(0,1)$
- **D** The empty set
- **E** $[0,1]$ union $[2,3]$

2.3 In $X = [0, 2]$ with its relative topology, find the interior, closure and boundary of $A = [0, 1)$. Is A open or closed in X ?

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Exercises 2.2-2.3

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2.2 C. An open interval is connected but not closed in $\mathbb{R}$, so it is not compact. The other options are compact or disconnected. The finite two-point set and the disjoint closed intervals are disconnected; the closed interval and the empty set are compact. The empty set is connected under the usual convention but still fails noncompactness.

2.3 In $X = [0, 2]$ with its relative topology, find the interior, closure and boundary of $A = [0, 1)$. Is A open or closed in X ? [4]

2.3 $A = X \cap (-1, 1)$ is open in X , so its relative interior is A . Its closure is $[0, 1]$ and its boundary is $\{1\}$. Zero is interior relative to X because a sufficiently small ball in X has no negative part. A is not closed in X : the sequence $1 - 1/n$ for $n \geq 2$ tends to the missing point 1. In the ambient real line, interior and boundary would instead be $(0, 1)$ and $\{0, 1\}$.

Exercise 3.1

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Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.