

Exercise walk-through

13.1

Sequences, series and uniform convergence

GRE · GRE Subject Test · GRE Mathematics

8. Choose and justify a method

Choose and justify a method

Pointwise convergence chooses an index N separately for each x and tolerance. Uniform convergence chooses one N that works for every x in the domain. For real-valued functions, check the supremum of $|f_n - f|$ over the whole domain. A continuous pointwise limit does not by itself prove uniform convergence. Domain endpoints and shrinking peaks often distinguish the two notions.

9. Choose and justify a method

Choose and justify a method

A uniform limit of continuous functions is continuous. On a closed bounded interval, uniform convergence of Riemann-integrable functions permits exchanging limit and integral. Exchanging derivatives needs extra hypotheses; uniform convergence of the functions alone is insufficient. The Weierstrass M-test establishes uniform absolute convergence of a series of functions if each term is bounded by M_n on the whole domain and $\sum M_n$ converges.

14. Reasoning and check

Reasoning and check

On $[0, 1]$, the limit is zero for $x < 1$ and one at 1. This limit is discontinuous, so continuous f_n cannot converge uniformly. More directly the supremum error on $[0, 1)$ is one. On $[0, a]$ the supremum is $a^n \rightarrow 0$, so convergence is uniform to zero. The p-series with $p=2$ converges absolutely. The alternating harmonic series converges by decreasing terms tending to zero, but the absolute series diverges; its convergence is conditional.

Exercise 2.1

2.1 Which statement must hold for a bounded real sequence?

[1]

- **A** It converges
- **B** It is monotone
- **C** It has a convergent subsequence
- **D** It is Cauchy
- **E** Its terms tend to zero

Exercise 2.1

2.1 Which statement must hold for a bounded real sequence?

[1]

- **A** It converges
- **B** It is monotone
- **C** It has a convergent subsequence
- **D** It is Cauchy
- **E** Its terms tend to zero

2.1 C. Bolzano–Weierstrass gives a convergent subsequence. The sequence $(-1)^n$ rules out the other necessary claims. The bounded sequence $(-1)^n$ is neither convergent, monotone nor Cauchy and does not tend to zero. Its even subsequence does converge.

Exercises 2.2·2.3

2.2 For $f_n(x) = \sin(nx)/n$ on $\mathbb{R}$, convergence to zero is what type?

[1]

- **A** Not pointwise
- **B** Pointwise but not uniform
- **C** Uniform
- **D** Only at integer x
- **E** Divergent at zero

2.3 Determine pointwise and uniform convergence of $f_n(x) = x^n$ on $[0, 1]$ and on $[0, a]$

Exercises 2.2·2.3

2.2 For $f_n(x) = \sin(nx)/n$ on $\mathbb{R}$, convergence to zero is what type? [1]

- **A** Not pointwise
- **B** Pointwise but not uniform
- **C** Uniform
- **D** Only at integer x
- **E** Divergent at zero

2.2 C. The supremum of the absolute value is $1/n$, independent of x , which tends to zero. A, D and E fail because the uniform error bound includes every real x , including zero and noninteger inputs. B fails because the same bound $1/n$ tends to zero independently of x .

2.3 Determine pointwise and uniform convergence of $f_n(x) = x^n$ on $[0, 1]$ and on $[0, a]$

2.3 On $[0, 1]$, the limit is zero for $x < 1$ and one at 1. This limit is discontinuous, so continuous f_n cannot converge uniformly. More directly the supremum error on $[0, 1)$ is one. On $[0, a]$ the supremum is $a^n \rightarrow 0$, so convergence is uniform to zero. The p -series with $p=2$ converges absolutely. The alternating harmonic series converges by decreasing terms tending to zero, but the absolute series diverges; its convergence is conditional.

Exercise 3.1

3.1 Let $g_n(x) = \sin(nx)/n$ on $\mathbb{R}$. Is convergence uniform? May its derivatives converge to the derivative of its limit? Use an explicit point to decide. [4]

Exercise 3.1

3.1 Let $g_n(x) = \sin(nx)/n$ on $\mathbb{R}$. Is convergence uniform? May its derivatives converge to the derivative of its limit? Use an explicit point to decide. [4]

3.1 $\sup_{x \in \mathbb{R}} |g_n(x)| = 1/n \rightarrow 0$, so the functions converge uniformly to zero. But $g'_n(x) = \cos(nx)$, and at $x = 0$ these derivatives are all 1. The derivative of the zero limit is 0. Thus uniform convergence of the functions alone does not justify differentiating a limit; additional derivative-convergence hypotheses are needed.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.