

Exercise walk-through

11.1

Polynomials and field extensions

GRE · GRE Subject Test · GRE Mathematics

6. Choose and justify a method

Choose and justify a method

A polynomial is irreducible over a field if it has positive degree and no factorisation into polynomials of smaller positive degrees there. A quadratic or cubic is irreducible exactly when it has no root in that field. This root test alone fails for degree four or higher; for instance $(x^2+1)(x^2+2)$ has no real root but is reducible over $\mathbb{R}$. Always name the base field.

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For nested finite-degree fields K inside L inside M , the tower law gives $[M:K]=[M:L][L:K]$. The degree of an algebraic element is the degree of its minimal polynomial. A finite extension of degree two does not contain an element of degree three over the base field. Over $\mathbb{Q}$, $\sqrt{2}$ has degree two; adjoining $\sqrt{3}$ as well produces a degree-four extension, since $\sqrt{3}$ is not in $\mathbb{Q}(\sqrt{2})$. Over $\mathbb{C}$, primitive n th roots of unity have exact order n and are the roots of the cyclotomic polynomial Φ_n .

17. Reasoning and check

Reasoning and check

Direct multiplication gives $x^4 + 4 = (x^2 - 2x + 2)(x^2 + 2x + 2)$, so it is reducible over $\mathbb{Q}$ despite having no real, hence no rational, roots. The no-root test characterises irreducibility only for degree two or three over a field. If an irreducible cubic root β belonged to a degree-two extension L , then $\mathbb{Q} \subset \mathbb{Q}(\beta) \subset L$ and the tower law would make 3 divide 2. This is impossible. The conclusion uses irreducibility, not merely a cubic equation.

Exercise 2.1

2.1 Which polynomial is irreducible over F_2 ?

[1]

- **A** x^2
- **B** x^2+1
- **C** x^2+x
- **D** x^2+x+1
- **E** x^3+x^2

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2.1 D. The quadratic x^2+x+1 has no root among 0 and 1. The others factor as $x \cdot x$, $(x+1)^2$, $x(x+1)$ and $x^2(x+1)$, respectively, over F_2 . The four rejected polynomials factor respectively as x times x , $(x+1)$ squared, $x(x+1)$, and x squared times $(x+1)$ in characteristic two.

Exercises 2.2·2.3

2.2 If $[L:K]=3$ and $[M:L]=2$, what is $[M:K]$?

[1]

- **A** 1
- **B** 2
- **C** 3
- **D** 5
- **E** 6

2.3 Construct $\mathbb{F}_2[x]/(x^2 + x + 1)$. List its elements and find the inverse of the residue

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- A 1
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- E 6

2.2 E. The tower law multiplies finite degrees, giving 2 times 3=6. One, two and three omit a nontrivial extension factor; five adds degrees instead of multiplying them.

2.3 Construct $\mathbb{F}_2[x]/(x^2 + x + 1)$. List its elements and find the inverse of the residue

2.3 The quadratic takes value one at both elements of $\mathbb{F}_2$, so it has no root and is irreducible. Its quotient is a field. Division leaves a representative of degree below two: $0, 1, \alpha, 1 + \alpha$. The relation is $\alpha^2 = \alpha + 1$ in characteristic two. Consequently $\alpha(\alpha + 1) = \alpha^2 + \alpha = 1$, so $\alpha^{-1} = \alpha + 1$. This is not $\mathbb{Z}/4\mathbb{Z}$, which has a nonzero zero divisor.

Exercise 3.1

3.1 Is $x^4 + 4$ irreducible over $\mathbb{Q}$? Does the absence of rational roots answer that question? Can a degree-two extension of $\mathbb{Q}$ contain a root of an irreducible cubic over $\mathbb{Q}$? [4]

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3.1 Direct multiplication gives $x^4 + 4 = (x^2 - 2x + 2)(x^2 + 2x + 2)$, so it is reducible over $\mathbb{Q}$ despite having no real, hence no rational, roots. The no-root test characterises irreducibility only for degree two or three over a field. If an irreducible cubic root β belonged to a degree-two extension L , then $\mathbb{Q} \subset \mathbb{Q}(\beta) \subset L$ and the tower law would make 3 divide 2. This is impossible. The conclusion uses irreducibility, not merely a cubic equation.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.