

Exercise walk-through

10.1

Rings, ideals and modules

GRE · GRE Subject Test · GRE Mathematics

6. Choose and justify a method

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For the domain, ideal and quotient examples below, use a commutative ring with identity 1 distinct from 0. A source problem may specify a general ring instead; do not assume its multiplication commutes unless stated or proved. A unit has a multiplicative inverse. A nonzero zero divisor multiplies some nonzero element to zero. An integral domain has no such zero divisors; cancellation of a nonzero factor then works. A field is a domain in which every nonzero element is a unit. $\mathbb{Z}$ is a domain but not a field; $\mathbb{Z}/6\mathbb{Z}$ is neither.

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A submodule is closed under addition and all permitted scalar actions. A linear map of modules preserves both. The kernel and image are submodules, and the quotient by the kernel is isomorphic to the image. Do not apply finite-dimensional rank-nullity to an arbitrary module without establishing an appropriate free-module setting; integer row operations and field row operations permit different divisions. In a general Boolean ring, every a satisfies $a^2=a$. Do not assume commutativity to prove it: idempotence of $a+a$ gives $4a=2a$, hence $2a=0$.

17. Reasoning and check

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A nonzero free $\mathbb{Z}$ -module has no nonzero vector annihilated by a nonzero integer: examine each integer coordinate. But $6[1] = 0$ in $\mathbb{Z}/6\mathbb{Z}$. It is not free; a field-style basis argument is invalid. $\mathbb{Z}/(2) \cong \mathbb{F}_2$ is a field, so (2) is maximal and prime. Evaluation at zero maps $\mathbb{Q}[x]$ onto $\mathbb{Q}$ with kernel (x) ; its quotient is a field too. Both arguments use proper ideals in commutative rings with identity.

Exercise 2.1

2.1 How many units are in $\mathbb{Z}/10\mathbb{Z}$?

[1]

- A 2
- B 3
- C 4
- D 5
- E 9

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2.1 C. Only 1,3,7,9 are coprime to 10; each has a multiplicative inverse. There are four units. Two or three omit units; five or nine include a residue divisible by two or five.

Exercises 2.2·2.3

2.2 Which integer ideal is maximal in $\mathbb{Z}$?

[1]

- A $0\mathbb{Z}$
- B $4\mathbb{Z}$
- C $6\mathbb{Z}$
- D $7\mathbb{Z}$
- E $9\mathbb{Z}$

2.3 In $\mathbb{Z}/12\mathbb{Z}$, list the units and nonzero zero divisors. Is the ideal (4) prime or maximal?[4]

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2.2 D. $\mathbb{Z}/7\mathbb{Z}$ is a field because 7 is prime. The composite quotients have zero divisors, and $\mathbb{Z}/0\mathbb{Z}$ is $\mathbb{Z}$, not a field. The zero ideal has quotient $\mathbb{Z}$, which is not a field. Quotients by four, six and nine have nonzero zero divisors, so those ideals are not maximal.

2.3 In $\mathbb{Z}/12\mathbb{Z}$, list the units and nonzero zero divisors. Is the ideal (4) prime or maximal?[4]

2.3 Units are the residues coprime to 12: 1, 5, 7, 11. The remaining nonzero residues 2, 3, 4, 6, 8, 9, 10 are zero divisors. For each such a , $a \cdot (12/\gcd(a, 12)) = 0$ modulo 12, with the second factor nonzero. The ideal $(4) = \{0, 4, 8\}$ gives quotient isomorphic to $\mathbb{Z}/4\mathbb{Z}$. This is not a domain, since $2 \cdot 2 = 0$, and is not a field. Hence the ideal is neither prime nor maximal.

Exercise 3.1

3.1 As a $\mathbb{Z}$ -module, can $\mathbb{Z}/6\mathbb{Z}$ be free? Compare the ideals (2) and (x) in $\mathbb{Z}$ and $\mathbb{Q}[x]$, respectively, using their quotients. [4]

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Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.