

Exercise walk-through

1.1

Single-variable calculus and applications

GRE · GRE Subject Test · GRE Mathematics

6. Choose and justify a method

Choose and justify a method

A limit describes values near a point, without requiring the function to be defined there. Continuity adds the requirement that the function value exists and agrees with the limit. Algebraic cancellation is valid only away from the cancelled zero, but can reveal a removable limit. One-sided limits must agree for a two-sided limit. For quotient limits, check the denominator and hypotheses before applying a rule; $0/0$ is an indeterminate form, not an answer.

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A derivative is a limit of difference quotients and implies continuity; the converse fails, as $|x|$ at zero shows. Product, quotient and chain rules describe different structures: the derivative of $f(g(x))$ is $f'(g(x))g'(x)$, not a product of unrelated values. Differentiating a composition a second time generally produces two terms. Check domains, nonzero denominators and differentiability assumptions before using a symbolic expression as a derivative.

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Choose and justify a method

The fundamental theorem says that the derivative of $\int$ from a to x of a continuous integrand $f(t)$ is $f(x)$. With a variable upper bound $h(x)$, multiply by $h'(x)$; with two variable bounds, subtract the corresponding lower-bound contribution. A definite integral also arises as a limit of Riemann sums. Rewrite the sum as $(1/n)\sum f(k/n)$ before identifying the integral on $[0,1]$, rather than treating n -dependent terms as constants.

Exercises 2.1·2.2

2.1 $f(x)=|x|$ at 0 is:

[1]

- **A** continuous and differentiable
- **B** continuous but not differentiable
- **C** neither

2.2 The series $\sum 1/n$ for $n \geq 1$:

[1]

- **A** converges because terms tend to zero
- **B** diverges
- **C** has finitely many terms

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2.1 $f(x)=|x|$ at 0 is: [1]

- **A** continuous and differentiable
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2.1 B. Left and right derivatives are -1 and 1 . A fails because one-sided derivatives differ; C fails because the function value and limit are both zero.

2.2 The series $\sum 1/n$ for $n \geq 1$: [1]

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2.2 B. The harmonic series diverges; the term test alone is insufficient. A confuses a necessary condition with a sufficient one; C fails because there is a term for every positive integer.

Exercise 2.3

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2.3 The integrand is continuous on the real line. The fundamental theorem and chain rule apply at both bounds:

$$F'(x) = (1 + x^4)2x - (1 + x^2).$$

$$F'(2) = (1 + 2^4)2 \cdot 2 - (1 + 2^2) = 63.$$

Independently, an antiderivative is $t + t^3/3$. Thus $F = x^2 + x^6/3 - x - x^3/3$, with the same derivative.

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3.1 Rewrite each term as $\frac{1}{n} \frac{k/n}{1+(k/n)^2}$. The continuous function $f(t) = t/(1+t^2)$ on $[0, 1]$ and mesh $1/n$ give

$$L = \int_0^1 \frac{t}{1+t^2} dt = \frac{1}{2} \ln 2.$$

The harmonic series diverges: the block from $2^{j-1} + 1$ through 2^j contains 2^{j-1} terms at least $1/2^j$, so each block adds at least $1/2$. Terms tending to zero is necessary but insufficient. The finite-sum integrand here depends on the scaled index; it is not the harmonic series.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.