

Revision

GRE · GRE Subject Test · GRE Mathematics

Apply the course version boundaries in notes.md; HSK tasks are transferable language practice, not a claimed HSK 3.0 mock.

gre mathematics · mixed-revision · 1

A linear map on $\mathbb{R}^3$ has rank 2. Find its nullity and explain what this means.

Worked answer

Nullity 1 by rank-nullity. Its kernel is a one-dimensional subspace.

gre mathematics · mixed-revision · 2

Evaluate the counterclockwise integral of $4/(z+1)$ on $|z+1|=2$.

Worked answer

$8\pi i$: one enclosed simple pole at -1 has residue 4.

gre mathematics · mixed-revision · 3

Solve $y' = -2y$ with $y(0) = 5$ and verify the result by differentiation.

Worked answer

$y = 5e^{-2t}$; derivative is $-10e^{-2t} = -2y$ and initial value is 5.

gre mathematics · mixed-revision · 4

In $\mathbb{Z}/24\mathbb{Z}$ under addition, find the order of 9 and list the subgroup it generates. How many cosets does it have?

Worked answer

The order is $24/\gcd(9,24)=8$. The subgroup is $\{0,3,6,9,12,15,18,21\}$; it has index $24/8=3$.

gre mathematics · mixed-revision · 5

The additive map $\mathbb{Z}/15\mathbb{Z}$ to $\mathbb{Z}/15\mathbb{Z}$ sends x to $5x$. Find its kernel and image and explain why the quotient by the kernel is not the entire codomain.

Worked answer

The kernel is $\{0, 3, 6, 9, 12\}$, the image is $\{0, 5, 10\}$, and the quotient has three elements. The codomain has fifteen elements; the map is not onto.

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In $\mathbb{Z}/9\mathbb{Z}$, list the units and nonzero zero divisors. Explain why the ring is not a field despite having an identity.

Worked answer

Units are 1, 2, 4, 5, 7, 8. Nonzero zero divisors are 3 and 6. Since $3 \cdot 3 = 0$ modulo 9 and 3 has no inverse, not every nonzero element is a unit.

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Compare $2\mathbb{Z}$ as a submodule of the $\mathbb{Z}$ -module $\mathbb{Z}$ with $2\mathbb{Z}$ as a proposed subspace of $\mathbb{Q}$. Give a scalar action that distinguishes them.

Worked answer

$2\mathbb{Z}$ is closed under integer scaling and addition, so it is a $\mathbb{Z}$ -submodule. It is not a $\mathbb{Q}$ -vector subspace because rational scalar $1/2$ sends 2 to 1, which is not in $2\mathbb{Z}$.

gre mathematics · mixed-revision · 8

Decide whether $x^2 + 1$ is irreducible over $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{F}_2$, explaining each change of coefficient field.

Worked answer

Over $\mathbb{Q}$ and $\mathbb{R}$ it has no root and is an irreducible quadratic. Over $\mathbb{F}_2$ it is $(x+1)^2$ and is reducible. The real field differs from the complex field, over which it splits.

gre mathematics · mixed-revision · 9

A finite field extension L/K has degree 10. Can an element of L have minimal polynomial of degree 3 over K ?

Worked answer

No. The intermediate field $K(\alpha)$ would have degree 3, and the tower law would require 3 to divide 10.

gre mathematics · mixed-revision · 10

Solve $8x \equiv 4 \pmod{12}$, listing every residue solution in the range 0 through 11.

Worked answer

$\gcd(8,12)=4$, so reduction gives $2x \equiv 1 \pmod{3}$. Thus $x \equiv 2 \pmod{3}$ and the four original residues are 2,5,8,11.

gre mathematics · mixed-revision · 11

Find the least nonnegative solution to $x \equiv 3 \pmod{5}$ and $x \equiv 4 \pmod{7}$. State the modulus of uniqueness.

Worked answer

Write $x=3+5k$; $5k \equiv 1 \pmod{7}$ gives $k \equiv 3$. Therefore $x \equiv 18 \pmod{35}$; 18 is the least nonnegative solution.

gre mathematics · mixed-revision · 12

Determine whether $f_n(x)=x/n$ converges uniformly to zero on $[0,2]$ and on $\mathbb{R}$.

Worked answer

On $[0,2]$, the supremum error is $2/n$, so convergence is uniform. At each fixed real x the limit is still zero, but on $\mathbb{R}$ the error is unbounded for every n , so convergence is not uniform there.

gre mathematics · mixed-revision · 13

Use the Weierstrass M-test to decide uniform convergence of $\sum \sin(nx)/n^2$ on $\mathbb{R}$. Explain why replacing n^2 by n defeats this argument.

Worked answer

Each absolute term is at most $1/n^2$ and that numerical series converges, so the function series converges uniformly and absolutely. The bound $1/n$ has a divergent sum, making this M-test inconclusive; that failure alone is not a proof of divergence.

gre mathematics · mixed-revision · 14

Find the interior, closure and boundary of the rationals $\mathbb{Q}$ as a subset of $\mathbb{R}$.

Worked answer

The interior is empty because every interval contains irrational points; the closure is $\mathbb{R}$ because rationals are dense. Thus the boundary is all of $\mathbb{R}$.

gre mathematics · mixed-revision · 15

In the space $X=(0,1)$, consider the whole set X . Is it relatively closed and relatively open? Is it compact in its usual metric?

Worked answer

The whole space is both open and closed relative to itself. It is not compact; the sequence $1/n$ for $n \geq 2$ has no subsequence converging to a point in X . Relative closedness alone does not provide Euclidean closedness in $\mathbb{R}$.

gre mathematics · mixed-revision · 16

Find the radius and complete real interval of convergence of $\sum x^n/(n \cdot 2^n)$, $n \geq 1$.

Worked answer

Radius 2. At -2 the alternating harmonic series converges, while at 2 the harmonic series diverges; the interval is $[-2, 2)$.

gre mathematics · mixed-revision · 17

Give the degree-three Maclaurin polynomial for $\sin x$ and an absolute error bound for $|x| \leq 0.2$ using a degree-three Taylor remainder.

Worked answer

The polynomial is $x - x^3/6$. The fourth derivative is $\sin x$ and has absolute value at most 1, so the Lagrange bound is $|x|^4/24 \leq 0.000066667$. A sharper bound is possible, but is not needed to justify this one.

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Determine whether the integral of $1/x^2$ from -1 to 1 exists as an ordinary improper integral or as a finite symmetric principal value.

Worked answer

Both fail. The integral diverges positively on each side of zero; symmetric cutoffs give $2/\epsilon - 2$, which tends to infinity rather than cancelling.

gre mathematics · mixed-revision · 19

Rotate the region between $y=2$ and $y=x$ on $0 \leq x \leq 2$ around the x -axis. Find its volume using washers.

Worked answer

Outer radius 2 and inner radius x give $V = \pi \int_0^2 (4 - x^2) dx = \pi(8 - 8/3) = 16\pi/3$.

gre mathematics · mixed-revision · 20

Classify the origin for $f=x^2+4xy+y^2$ and $g=x^4+y^4$, explaining which Hessian test is decisive.

Worked answer

For f , $f_{xx}=f_{yy}=2$ and $f_{xy}=4$, so $D=4-16=-12$; it is a saddle. For g the Hessian test is inconclusive, but $g \geq 0$ with equality only at the origin proves a strict global minimum.

gre mathematics · mixed-revision · 21

Find the maximum and minimum of $x+y$ on $x^2+y^2=8$, stating the points where they occur.

Worked answer

$(x+y)^2 \leq 2(x^2+y^2)=16$, so the range is $[-4,4]$. Maximum 4 occurs at $(2,2)$, minimum -4 at $(-2,-2)$. The compact circle ensures extrema are attained.

gre mathematics · mixed-revision · 22

Use the divergence theorem to compute the outward flux of $F=(x,2y,3z)$ through the unit cube $0 \leq x,y,z \leq 1$.

Worked answer

$\operatorname{div} F=1+2+3=6$. The closed cube has volume 1, so total outward flux is 6. Opposite face orientations must be respected in a direct calculation.

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Evaluate the clockwise unit-circle circulation of $F=(-y,x)$. Explain why Green's theorem cannot be applied over the full disk to $F=(-y/(x^2+y^2), x/(x^2+y^2))$.

Worked answer

For the first field, counterclockwise circulation is 2π ; clockwise is -2π . The second field is undefined at the origin and lacks the smoothness hypothesis on the full disk, even though its curl vanishes away from zero.

gre mathematics · mixed-revision · 24

For $f(x)=3x-4$ on $\mathbb{R}$, find its inverse and decide whether f is an involution.

Worked answer

The inverse is $(y+4)/3$. It is not an involution: $f(f(x))=9x-16$ is not identically x .

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Let $g(x)=2/x$ on $\mathbb{R}$ excluding zero. Verify the inverse composition and evaluate $g^9(4)$.

Worked answer

$g(g(x))=2/(2/x)=x$ with every input nonzero. Odd iterates equal g , so $g^9(4)=1/2$.

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Give a function and disjoint nonempty input sets whose images intersect. State a condition on the function that rules this out.

Worked answer

Use $f(x)=x^2$, $A=\{-3\}$, $B=\{3\}$; both images are $\{9\}$. Injectivity ensures equal outputs require equal inputs, so disjoint input sets then have disjoint images.

gre mathematics · mixed-revision · 27

Negate the claim that every integer has an integer strictly larger than it. Distinguish the negation from a counterexample to the original claim.

Worked answer

The negation is: there exists an integer x such that every integer y satisfies $y \leq x$. The original is true because $y=x+1$ is always an integer; its negation is false, and no counterexample to the original exists.

gre mathematics · mixed-revision · 28

$P(A)=0.6$, $P(B)=0.4$ and $P(A \cap B)=0.24$. Find both conditional probabilities and test independence.

Worked answer

$P(A|B)=0.6$; $P(B|A)=0.4$. The joint probability is the product $0.6 \cdot 0.4$, so the events are independent.

gre mathematics · mixed-revision · 29

A sample mean has standard error 4 at $n=9$ for iid observations. How large must n be to reduce standard error to 1?

Worked answer

Population standard deviation is $4\sqrt{9}=12$. Solve $12/\sqrt{n}=1$ to obtain $n=144$; a fourfold precision gain needs sixteen times as many observations.

gre mathematics • mixed-revision • 30

A similar model uses 80% of every original length. Find the area and volume decreases as percentages.

Worked answer

Length factor 0.8 gives area factor 0.64 and volume factor 0.512. The decreases are 36% and 48.8%, respectively.

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Foci are $(-5,0), (5,0)$. Classify the locus $r_{\text{left}} + r_{\text{right}} = 14$ and the locus $r_{\text{left}} - r_{\text{right}} = -6$; specify any branch.

Worked answer

The first is an ellipse with $a=7, c=5, b^2=49-25=24$: $x^2/49 + y^2/24 = 1$. The second is a hyperbola $x^2/9 - y^2/16 = 1$ restricted to $x \leq -3$, because the signed difference is negative.

gre mathematics • mixed-revision • 32

For $y = 3 \cos(4x - \pi)$, give the amplitude, period and horizontal shift. Find one maximum location.

Worked answer

Amplitude 3; period $\pi/2$; right shift $\pi/4$. One maximum is $x = \pi/4$, where the argument is zero.

gre mathematics • mixed-revision • 33

For $x=t, y=t^2$ with t in $[-1, 1]$, give the traced set, direction and slope at $t = -1/2$.

Worked answer

Only the parabola segment $y=x^2$, $-1 \leq x \leq 1$, is traced. It moves from $(-1, 1)$ through $(0, 0)$ to $(1, 1)$. The slope is $2t$, hence -1 at $t = -1/2$.

gre mathematics • mixed-revision • 34

Find the projection of $(2, 1)$ onto the line spanned by $(1, 1)$, and verify its remainder is orthogonal.

Worked answer

Coefficient is $(2+1)/(1+1) = 3/2$, so projection is $(3/2, 3/2)$. Remainder $(1/2, -1/2)$ has dot product zero with $(1, 1)$.

gre mathematics · mixed-revision · 35

Compute signed and unsigned volumes for edges $(1,0,0), (0,2,0), (0,0,3)$. What changes if the last two edges are exchanged?

Worked answer

The scalar triple product is 6 and volume is 6. Exchanging the last two edges changes the signed result to -6 , while geometric volume remains 6.

gre mathematics · mixed-revision · 36

Solve $y' + 2y = 6$ with $y(0) = 1$ and check the equation.

Worked answer

Integrating factor e^{2x} gives $y = 3 + C e^{-2x}$; $C = -2$, so $y = 3 - 2e^{-2x}$. Its derivative is $4e^{-2x}$, and $y' + 2y = 6$.

gre mathematics · mixed-revision · 37

Give the general solution of $y'' - 2y' + y = e^x$, including the resonant particular term.

Worked answer

The repeated characteristic root is 1, giving $(C_1 + C_2 x)e^x$. A particular solution is $x^2 e^x / 2$, because $(D-1)^2(e^x z) = e^x z''$ and $z'' = 1$. Add both contributions.

gre mathematics · mixed-revision · 38

Count allocations of 10 identical tokens to 3 named boxes when the first gets at least 2 and the others may be empty.

Worked answer

Subtract the first-box lower bound 2. There are 8 remaining stars and 2 bars, so $C(10, 2) = 45$ allocations.

gre mathematics · mixed-revision · 39

Explain why $S = m^2$ is preserved when S increases by B and then B increases by 2, starting $S = 0, B = 1$. Give the state after four iterations.

Worked answer

Before an iteration $B = 2m + 1$. Updating S gives $m^2 + 2m + 1 = (m + 1)^2$, then B becomes $2(m + 1) + 1$. After four iterations $S = 16, B = 9$; B is the next addend, not the last one.

gre mathematics · mixed-revision · 40

For $x=u+2v, y=3u+v$, compute v_x and u_y .

Worked answer

$J=\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ has determinant -5 ; inverse is $\begin{bmatrix} -1/5 & 2/5 \\ 3/5 & -1/5 \end{bmatrix}$. Therefore $v_x=3/5$ and $u_y=2/5$.

gre mathematics · mixed-revision · 41

Why does the map $(u,v) \mapsto (u^3, v)$ contradict the claim that zero Jacobian determinant means no inverse exists?

Worked answer

It is bijective on $\mathbb{R}^2$ with inverse $(x,y) \mapsto (\text{real cube root of } x, y)$. Its determinant $3u^2$ vanishes at $u=0$, where the inverse fails to be differentiable. Nonzero determinant is a sufficient differentiable-local-inverse condition, not a necessary condition for mere invertibility.

gre mathematics · mixed-revision · 42

For $L=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $U=\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$, $b=(7,20)$, solve $LUx=b$.

Worked answer

Forward $y_1=7, y_2=20-14=6$. Backward $x_2=3, x_1=(7-3)/3=4/3$. Verify $Ux=(7,6)$ and $Ly=(7,20)$.

gre mathematics · mixed-revision · 43

Maps $A, B: \mathbb{R}^5 \rightarrow \mathbb{R}^5$ have nullities 3 and 1, respectively. Find bounds on nullity AB and explain why it cannot be 1.

Worked answer

Rank $B=4$. The intersection of $\text{im } B$ (dimension 4) with $\ker A$ (dimension 3) has dimension at least $4+3-5=2$ and at most 3. Therefore nullity AB ranges from $1+2=3$ to $1+3=4$; the ambient dimension forces overlap.

gre mathematics · mixed-revision · 44

Reverse and evaluate $\int_0^1 \int_x^1 e^{\widehat{(2y)}} dy \, dx$.

Worked answer

Region $0 \leq x \leq y \leq 1$ gives $\int_0^1 y e^{(2y)} dy$. Integration by parts gives $[y e^{(2y)}/2 - e^{(2y)}/4]_0^1 = (e^2 + 1)/4$. The factor y comes from the inner x -interval length.

gre mathematics • mixed-revision • 45

Evaluate $\int_0^\pi \sin(8x)/\sin x \, dx$ and justify treatment of both endpoints.

Worked answer

Reflection $x \mapsto \pi - x$ reverses the integrand sign. The endpoint limits are 8 and -8 , so singularities are removable and the integral is 0.

gre mathematics • mixed-revision • 46

A sphere has $R=2$, depth $h=1$ increasing at $1/2$ per time unit. Find the signed volume rate.

Worked answer

Current section area is $\pi(4-1)=3\pi$, so signed $dV/dt=3\pi/2$, positive for filling.

gre mathematics • mixed-revision • 47

For $f(x)=\sin(4x)$, $g(x)=5x$, give limits of f/g and $(f^2-f)/(2g-g^3)$ as $x \rightarrow 0$.

Worked answer

The first is $f'(0)/g'(0)=4/5$. The second is $-f'(0)/(2g'(0))=-2/5$ after factoring, with all nearby denominators checked.

gre mathematics • mixed-revision • 48

In the divisor basis on integers at least 2, compute closure of $\{12\}$ and test membership of 6 and 36.

Worked answer

Closure is $\{12, 24, 36, \dots\}$. The point 36 belongs; 6 does not, because U_6 excludes 12.

gre mathematics • mixed-revision • 49

For $d(x,y)=|x^3-y^3|$ on $\mathbb{R}$, is x_n equal to the cube root of n a Cauchy sequence? Explain using the image.

Worked answer

No. Its image is n , not a Cauchy sequence in ordinary $\mathbb{R}$. The space is complete, but completeness does not make every sequence Cauchy.

gre mathematics • mixed-revision • 50

Find the residue of $e^{2z}/(z-1)^3$ at $z=1$ and its positively oriented integral around a circle centred at 1 containing no other singularity.

Worked answer

For a triple pole, residue is $g''(1)/2! = 4e^2/2 = 2e^2$. The contour integral is $2\pi i$ times the residue, or $4\pi i e^2$.

gre mathematics • mixed-revision • 51

Count conjugacy classes in S_5 by listing the partitions of 5.

Worked answer

The partitions are 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1 and 1+1+1+1+1. There are seven classes. Orders alone do not distinguish every type.

gre mathematics • mixed-revision • 52

In a general ring where every element is idempotent, prove $2a=0$ without assuming that distinct elements commute.

Worked answer

Apply idempotence to $a+a$: $(a+a)^2 = a+a$. Expanding gives $4a^2 = 2a$; since $a^2 = a$, subtract $2a$ to obtain $2a=0$. Only distributivity and additive-ring operations were used.

gre mathematics • mixed-revision • 53

Primitive eighth roots are the roots of x^4+1 . Find their sum and product, and justify why these roots have exact order eight.

Worked answer

Their fourth power is -1 , so their eighth power is 1 and their order cannot divide 4. The divisors of 8 are 1,2,4,8, so their order is 8. Vieta gives sum 0 and product 1.