# GAC004 Mathematics I: Fundamentals

GAC Mathematics

## What this module is, and how it is marked

GAC004 rebuilds school mathematics in English so that you can study a university subject that uses it. The mathematics itself is familiar. The words, the symbols and the way a solution is written out are the new part, and they carry marks.

The module runs over six units and is assessed at your own teaching centre in several separate events, not in one final paper. A typical set is a short **in-class test** 课堂测验 on the early units, one or two **projects** 项目 built in a spreadsheet, an **examination** 考试 covering all six units, and **coursework** 平时作业 that includes a terminology logbook. Your centre publishes the exact weights; the point to take from the list is that marks accumulate all semester.

- The **terminology logbook** 术语记录本 is a real component. You write down each new mathematical word with a meaning **in your own words**, and it is inspected during the semester.
- Method marks exist. A correct answer with no working can score less than a wrong answer with a clear, correct method, so write the steps down.
- Read the **command word** 指令词 in the question. *Solve* wants a value, *simplify* wants a shorter expression, *analyse* wants a comment on what the answer means.

## Terminology and arithmetic review

The vocabulary of number is assumed by every later unit.

An **integer** 整数 is a whole number, positive or negative. A **rational number** 有理数 can be written as a fraction of two integers; an **irrational number** 无理数 cannot, and $\pi$ and $\sqrt{2}$ are the standard examples. A **prime number** 质数 has exactly two factors, itself and 1.

The **order of operations** 运算顺序 fixes what a written expression means: brackets, then indices, then multiplication and division, then addition and subtraction, working left to right within a level.

$$3 + 4 \times 2^2 = 3 + 4 \times 4 = 3 + 16 = 19$$

- A **factor** 因数 divides a number exactly; a **multiple** 倍数 is what you get by multiplying it. 6 is a factor of 24, and 24 is a multiple of 6.

- The **highest common factor (HCF)** 最大公因数 and **lowest common multiple (LCM)** 最小公倍数 come from the prime factors: $24 = 2^3 \times 3$ and $36 = 2^2 \times 3^2$, so the HCF is $2^2 \times 3 = 12$ and the LCM is $2^3 \times 3^2 = 72$.
- A **percentage** 百分比 is a fraction with denominator 100. An increase of 15% multiplies by 1.15; a decrease of 15% multiplies by 0.85. Reversing a percentage change means dividing, never subtracting the same percentage back.
- **Significant figures** 有效数字 report how precisely a value is known. 0.004 072 to three significant figures is 0.004 07.
- **Standard form** 科学记数法 writes a number as $a \times 10^n$ with $1 \leq a < 10$. It is how a calculator shows a very large or very small answer.

**Worked example.** A shop raises a price of 240 yuan by 15%, then reduces the new price by 15%. Is the final price 240 yuan?

$$240 \times 1.15 = 276, \qquad 276 \times 0.85 = 234.60$$

No. The two percentages act on different amounts, so the final price is 234.60 yuan. This is the most common percentage error in the module.

# Algebra I: introductory algebra

Algebra is arithmetic with letters standing for numbers, and it has its own vocabulary.

In $5x^2 - 3x + 7$, the whole thing is an **expression** 表达式; $5x^2$, $-3x$ and 7 are its **terms** 项; 5 is the **coefficient** 系数 of $x^2$; and 7 is a **constant** 常数.

An **equation** 方程 says two expressions are equal and is solved for a value. An **identity** 恒等式 is true for every value. An **inequality** 不等式 compares two expressions with $<$, $\leq$, $>$ or $\geq$.

To **expand** 展开 is to remove brackets; to **factorise** 因式分解 is to put them back.

$$(x+3)(x-5) = x^2 - 5x + 3x - 15 = x^2 - 2x - 15$$

- Solving a **linear equation** 一元一次方程 means doing the same operation to both sides until the letter stands alone.
- A **quadratic equation** 一元二次方程 has an $x^2$ term. Try factorising first; if that fails, use the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- **Simultaneous equations** 联立方程 are two equations in two unknowns, solved by substitution or by elimination. The solution is the point where the two lines meet.

- △ An inequality **reverses** 反向 when you multiply or divide both sides by a negative number: from $-2x > 6$ it follows that $x < -3$.

**Worked example.** Solve $x^2 - 2x - 15 = 0$.

$$x^2 - 2x - 15 = (x-5)(x+3) = 0$$

A product is zero only when a factor is zero, so $x = 5$ or $x = -3$. Substituting both back into the original equation is how you check, and it costs ten seconds.

# Algebra: graphs and coordinate geometry

A graph turns an equation into a picture, and coordinate geometry measures that picture.

A straight line is $y = mx + c$, where $m$ is the **gradient** 斜率 and $c$ the **y-intercept** y 轴截距. The gradient is the rise divided by the run.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- Two lines are **parallel** 平行 when their gradients are equal, and **perpendicular** 垂直 when the gradients multiply to $-1$.
- The **midpoint** 中点 of a segment is the average of the endpoints, and the **distance** 距离 between two points comes from Pythagoras: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- A quadratic graphs as a **parabola** 抛物线. Its **vertex** 顶点 is the turning point, and the **roots** 根 are where it crosses the $x$-axis, which are the solutions of $y = 0$.
- To solve two equations **graphically** 用图像求解, draw both and read the coordinates of the intersection. The answer is only as accurate as the drawing, which is why an algebraic check matters.

**Worked example.** Find the equation of the line through $(2, 1)$ and $(6, 9)$.

$$m = \frac{9-1}{6-2} = \frac{8}{4} = 2$$

Substituting $(2, 1)$ into $y = 2x + c$ gives $1 = 4 + c$, so $c = -3$ and the line is $y = 2x - 3$. Check the second point: $2(6) - 3 = 9$ ✓.

# Geometry: plane, solid and Euclidean geometry

Plane geometry is about flat shapes, solid geometry about three-dimensional ones, and Euclidean geometry is the reasoning that connects them.

Angles on a straight line add to 180°, angles around a point to 360°, and the interior angles of any triangle to 180°. In a **polygon** 多边形 of $n$ sides the interior angles add to $(n-2) \times 180°$.

- **Congruent** 全等 shapes are identical in size and shape; **similar** 相似 shapes have the same shape with all lengths in one ratio.
- **Pythagoras' theorem** 勾股定理 holds in a right-angled triangle: $a^2 + b^2 = c^2$, where $c$ is the **hypotenuse** 斜边.
- **Area** 面积 is measured in square units and **volume** 体积 in cubic units. A cylinder has volume $\pi r^2 h$; a sphere has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$.
- △ Scaling is not linear. If every length of a solid is doubled, its area is multiplied by $2^2 = 4$ and its volume by $2^3 = 8$.

**Worked example.** A cylindrical tank has radius 1.2 m and height 3.0 m. What volume does it hold, in litres?

$$V = \pi r^2 h = \pi \times 1.2^2 \times 3.0 = 13.57 \text{ m}^3$$

One cubic metre is 1000 litres, so the tank holds about 13,570 litres. Carrying the unit through the calculation is what makes the final conversion obvious.

# Trigonometry

Trigonometry connects the angles of a triangle to its sides.

In a right-angled triangle, with $\theta$ one of the acute angles, the three ratios are $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$, $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ and $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$.

For any triangle, the **sine rule** 正弦定理 and the **cosine rule** 余弦定理 apply:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \qquad a^2 = b^2 + c^2 - 2bc\cos A$$

- An angle can be measured in **degrees** 度 or in **radians** 弧度, where $\pi$ radians is 180°. Check which mode your calculator is in before every question.
- The **unit circle** 单位圆 extends the ratios beyond 90° and explains why sin and cos repeat every 360°: they are **periodic** 周期的.

- A **bearing** 方位角 is measured clockwise from north and always written with three figures, such as 075°.
- **Angles of elevation and depression** 仰角与俯角 are measured from the horizontal, upwards and downwards respectively.

**Worked example.** From a point 40 m from the base of a tower, the angle of elevation of the top is 32°. How tall is the tower?

$$\tan 32° = \frac{h}{40}, \qquad h = 40 \tan 32° = 25.0 \text{ m}$$

The opposite side and the adjacent side are involved, which is what selects the tangent ratio. Choosing the ratio is the marked step, not the arithmetic.

# Exponential and logarithmic functions

These two functions describe growth and decay, and they undo each other.

An **exponential function** 指数函数 is $y = a^x$: the variable is in the **exponent** 指数. It grows by a constant *factor* per step, which is why it eventually outgrows any straight line.

A **logarithm** 对数 answers the reverse question. $\log_a y = x$ means exactly $a^x = y$, so $\log_{10} 1000 = 3$.

$$\log(mn) = \log m + \log n, \qquad \log\left(\frac{m}{n}\right) = \log m - \log n, \qquad \log(m^k) = k \log m$$

- The laws turn multiplication into addition, which is what makes a logarithm useful for solving an equation with the unknown in the exponent.
- **Natural logarithms** 自然对数 use the base $e \approx 2.718$ and are written ln.
- **Compound interest** 复利 is exponential: $A = P(1 + r)^n$ after $n$ periods.
- **Exponential decay** 指数衰减 has a base between 0 and 1, and models cooling, medicine leaving the blood, and radioactive material.

**Worked example.** 5000 yuan is invested at 4% per year, compounded annually. How many whole years until it exceeds 7000 yuan?

$$5000(1.04)^n > 7000 \;\Rightarrow\; 1.04^n > 1.4 \;\Rightarrow\; n > \frac{\log 1.4}{\log 1.04} = 8.58$$

So it takes 9 whole years. Taking logarithms of both sides is the step that brings $n$ down from the exponent, and it is the technique the unit exists to teach.