

Exercise walk-through

4.5

Graphs and Networks

GAC Mathematics

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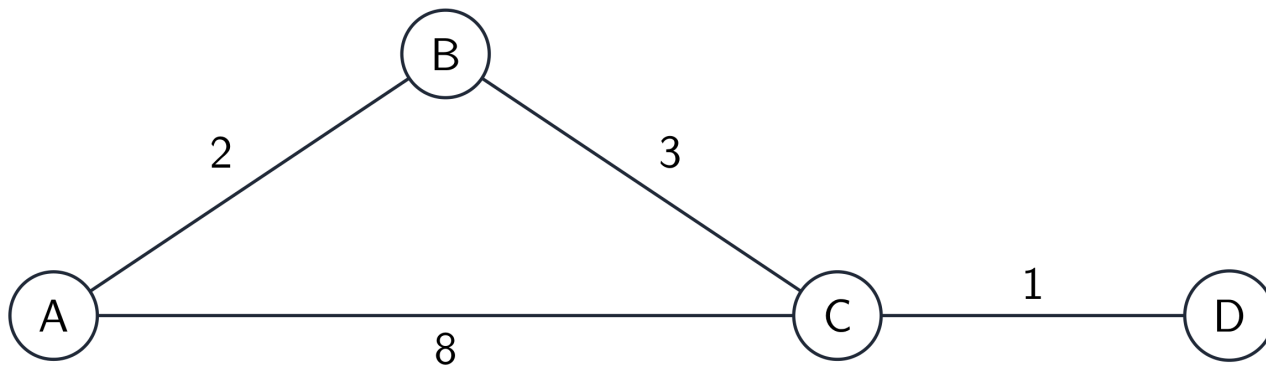
The method

Method — Key knowledge: state whether connections are directed, weighted or undirected

A graph has **vertices** connected by **edges**. This sheet uses finite simple undirected graphs unless a question says otherwise: no loops or repeated edges. Degree counts incident edges; the sum of degrees is twice the edge count. A path connects vertices, and a cycle returns to its start without repeating other vertices. A **tree** is connected and has no cycles; a tree with n vertices has $n - 1$ edges. That edge count alone does not prove a graph is a tree without connectedness. In a weighted network, a shortest path minimises total path weight, whereas a minimum spanning tree connects all vertices with minimum total edge weight and no cycle. These are different optimisation tasks.

Method — Worked example: compare total route weight, not the number of edges

Known edge weights are $AB = 2$, $BC = 3$, $AC = 8$ and $CD = 1$. The path A-C-D has weight 9, while A-B-C-D has weight 6. Therefore the three-edge path is shorter by weight despite having more edges. The minimum spanning tree for this small network uses AB, BC and CD with total 6; the agreement of totals here does not make the tasks identical.



Exercises 2.1-2.2

2.1 A simple undirected graph has edges AB, AC and BC. Find the degree of A and the total edge count. [2]

2.2 A tree has 9 vertices. Find its edge count and state the property excluding a triangle cycle. [2]

Exercises 2.1·2.2

2.1 A simple undirected graph has edges AB, AC and BC. Find the degree of A and the total edge count. [2]

2.1

A is incident to AB and AC, so degree 2. Total edge count 3.

2.2 A tree has 9 vertices. Find its edge count and state the property excluding a triangle cycle. [2]

2.2

A tree has $n - 1 = 9 - 1 = 8$ edges. It is acyclic, so no triangle cycle is allowed.

Exercises 2.3-2.4

2.3 A simple undirected graph has degrees 3, 2, 2, 1. Find its edge count and explain why degree sum 9 would be impossible. [3]

2.4 A graph has edges AB, BC, CA and DE, on exactly five vertices A to E. It has n minus 1 edges. Decide whether it is a tree and justify your decision using two graph properties. [3]

3 Exam-style questions

Exercises 2.3·2.4

2.3 A simple undirected graph has degrees 3, 2, 2, 1. Find its edge count and explain why degree sum 9 would be impossible. [3]

2.3

Each edge contributes two to total degree.

$$|E| = \frac{\sum d(v)}{2} = \frac{3 + 2 + 2 + 1}{2} = 4$$

Four edges. A sum of 9 is odd and cannot equal twice an integer edge count.

2.4 A graph has edges AB, BC, CA and DE, on exactly five vertices A to E. It has n minus 1 edges. Decide whether it is a tree and justify your decision using two graph properties. [3]

3 Exam-style questions

2.4

It is not a tree. The ABC triangle is a cycle. The DE component is disconnected from ABC. The four-edge count alone is insufficient.

Exercise 3.1

3.1 A weighted undirected network has edges $AB = 2$, $AC = 6$, $BC = 1$, $BD = 7$, $CD = 3$. (a) Find a shortest path from A to D and its weight. (b) Find a minimum spanning tree and its total weight. (c) Explain the different requirements of these tasks. [6]

Exercise 3.1

3.1 A weighted undirected network has edges $AB = 2$, $AC = 6$, $BC = 1$, $BD = 7$, $CD = 3$. (a) Find a shortest path from A to D and its weight. (b) Find a minimum spanning tree and its total weight. (c) Explain the different requirements of these tasks. [6]

3.1

(a) A-B-C-D has weight $2 + 1 + 3 = 6$. Correct path earns B1 and weight B1. The other simple A-to-D alternatives are A-B-D with 9, A-C-D with 9, and A-C-B-D with 14, so 6 is shortest.

(b) Choose BC, AB and CD to connect all vertices without a cycle. Total weight 6.

(c) The shortest-path task connects only the specified endpoints with minimum route weight. The spanning-tree task connects all vertices with no cycles and minimum total included edge weight.

Exercise 3.2

3.2 A weighted undirected network on A, B, C, D and E has edges $AB = 1$, $AC = 4$, $BC = 2$, $BD = 5$, $CD = 1$, $CE = 6$, $DE = 3$. (a) Find a minimum spanning tree and its total weight, showing how cycles are avoided. (b) Find a shortest path from A to E and its weight. (c) If DE closes, find a new shortest A-to-E route using the remaining edges. Explain why that new route must be checked rather than retaining the old total. [8]

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3.2 A weighted undirected network on A, B, C, D and E has edges $AB = 1$, $AC = 4$, $BC = 2$, $BD = 5$, $CD = 1$, $CE = 6$, $DE = 3$. (a) Find a minimum spanning tree and its total weight, showing how cycles are avoided. (b) Find a shortest path from A to E and its weight. (c) If DE closes, find a new shortest A-to-E route using the remaining edges. Explain why that new route must be checked rather than retaining the old total. [8]

3.2

(a) Select AB and CD, both weight 1. Add BC with weight 2, joining those components without a cycle. Add DE with weight 3 to connect E. Total $W = 1 + 1 + 2 + 3 = 7$, with all five vertices connected and four edges.

(b) Shortest path A-B-C-D-E. Weight $W_p = 1 + 2 + 1 + 3 = 7$. Nonnegative edge weights and comparison with alternative approaches to C/D establish no shorter route.

(c) E is now accessible only through CE, so the shortest remaining route is A-B-C-E with weight $1 + 2 + 6 = 9$. Correct route and weight. The closed edge cannot be used; the old path is infeasible and its total is not a valid remaining-network answer.