

Course Companion

GAC Mathematics

Every topic handout in one volume

exercises.lol

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GAC004 Mathematics I: Fundamentals

GAC Mathematics

What this module is, and how it is marked

A price can fall by 20% and then rise by 20% without returning to its starting value. Mathematics I helps you explain such results with a method, units and a clear interpretation. The six units connect arithmetic, algebra, graphs, geometry, trigonometry and exponential models.

Your centre's current assessment brief is the authority for tasks, weights, permitted tools and deadlines. An **in-class test** is completed under stated classroom conditions. **Projects** can involve collecting data, calculating and reporting; an **examination** and **coursework** may have different instructions. Do not infer an official assessment pattern from these practice sheets.

A **terminology logbook** can record the English term, a meaning in your own words and a small example. Follow your centre's instructions if it is submitted or assessed.

Read the **command word** : *solve* asks for values satisfying a condition, *simplify* for an equivalent expression, and *justify* for a reason. Show a valid method so a reader can check how the result follows. The mark scheme for each task determines its marks.

The six accompanying practice sheets are original GAC004-aligned material, not official papers. Their solutions award marks for stated mathematical steps rather than for the length of a written report.

Terminology and arithmetic review

The vocabulary of number is assumed by every later unit.

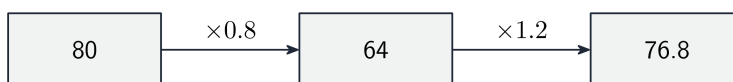
An **integer** is a whole number, including negative whole numbers and zero. A **rational number** can be written as a fraction of two integers with a nonzero denominator; an **irrational number** cannot, and π and $\sqrt{2}$ are the standard examples. A **prime number** is a positive integer greater than 1 with exactly two positive factors, itself and 1.

The **order of operations** fixes what a written expression means: brackets, then indices, then multiplication and division, then addition and subtraction, working left to right within a level.

$$3 + 4 \times 2^2 = 3 + 4 \times 4 = 3 + 16 = 19$$

- A **factor** divides a number exactly; a **multiple** is what you get by multiplying it. 6 is a factor of 24, and 24 is a multiple of 6.
- The **highest common factor (HCF)** and **lowest common multiple (LCM)** come from the prime factors: $24 = 2^3 \times 3$ and $36 = 2^2 \times 3^2$, so the HCF is $2^2 \times 3 = 12$ and the LCM is $2^3 \times 3^2 = 72$.
- A **percentage** is a fraction with denominator 100. An increase of 15% multiplies by 1.15; a decrease of 15% multiplies by 0.85. Reversing a percentage change means dividing, never subtracting the same percentage back.
- **Significant figures** are the digits retained to express a value at a stated precision. They do not alone establish the accuracy of a measurement. 0.004 072 to three significant figures is 0.004 07.
- **Standard form** writes a number as $a \times 10^n$ with $1 \leq |a| < 10$ for a nonzero number, with integer n. It is how a calculator shows a very large or very small answer.

Worked example. a decrease and increase use different bases



each multiplier uses the current amount

Known: an invented price of 80 units falls by 20%, then rises by 20%. Why use successive multipliers? Each change is calculated from the current price.

$$P_{new} = P_{old}(1 - r/100)$$

$$P_1 = 80(1 - 20/100) = 64$$

$$P_{new} = P_{old}(1 + r/100)$$

$$P_2 = 64(1 + 20/100) = 76.8$$

The price is not back at 80. The decrease was 16, while the later increase was 12.8, because the bases differ.

Continue with practice sheet **1.1**. Solve before opening the solutions, and check both the method and the final units.

Algebra I: introductory algebra

Algebra is arithmetic with letters standing for numbers, and it has its own vocabulary.

In $5x^2 - 3x + 7$, the whole thing is an **expression** ; $5x^2$, $-3x$ and 7 are its **terms** ; 5 is the **coefficient** of x^2 ; and 7 is a **constant** .

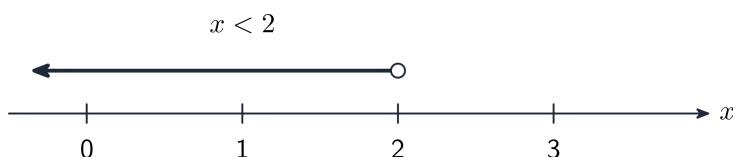
An **equation** says two expressions are equal and is solved for a value. An **identity** is true for every value in its domain. An **inequality** compares two expressions with $<$, $\leq$, $>$ or $\geq$.

To **expand** is to remove brackets; to **factorise** is to put them back.

$$(x + 3)(x - 5) = x^2 - 5x + 3x - 15 = x^2 - 2x - 15$$

- Solving a **linear equation** means doing the same operation to both sides until the letter stands alone.
- A **quadratic equation** has the form $ax^2 + bx + c = 0$ with $a \neq 0$.
Factorise when suitable, or use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- **Simultaneous equations** are two equations in two unknowns, solved by substitution or by elimination. For two linear equations, a unique solution is their intersection. Parallel distinct lines have no solution; identical lines have infinitely many.
- $\triangle$ An inequality **reverses** when you multiply or divide both sides by a negative number: from $-2x > 6$ it follows that $x < -3$.

Worked example. reverse the inequality sign when dividing by a negative



Known: solve $7 - 3x > 1$. Subtract 7 from both sides, then divide by negative 3. The last operation reverses the inequality.

$$7 - 3x > 1 \implies -3x > -6$$

$$x < \frac{-6}{-3} = 2$$

The open circle excludes 2. Check $x = 1$: $7 - 3(1) = 4 > 1$; the boundary $x = 2$ gives equality and is excluded.

Continue with practice sheet **1.2**. Solve before opening the solutions, and check both the method and the final units.

Algebra: graphs and coordinate geometry

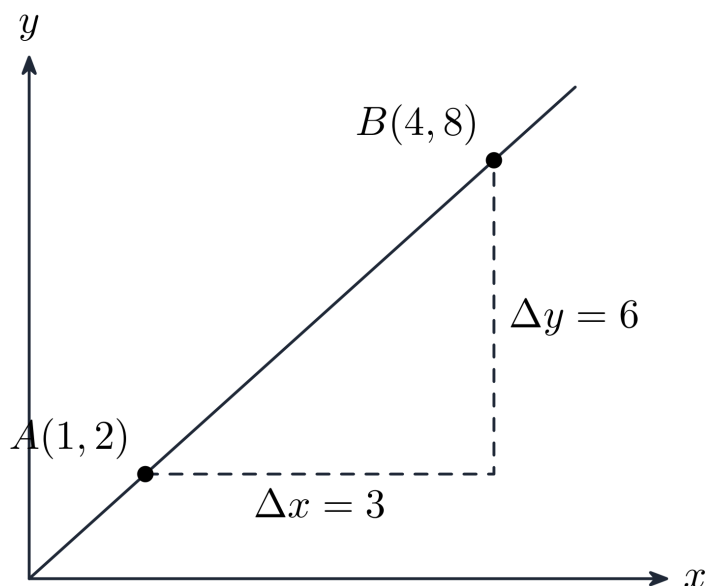
A graph turns an equation into a picture, and coordinate geometry measures that picture.

A nonvertical straight line can be written $y = mx + c$, where m is the **gradient** and c the **y-intercept**. The gradient is the rise divided by the run.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- Distinct nonvertical lines are **parallel** when their gradients are equal. Two nonvertical lines are **perpendicular** when their gradients multiply to -1 . Vertical lines have undefined gradient; a vertical and a horizontal line are perpendicular.
- The **midpoint** of a segment is the average of the endpoints, and the **distance** between two points comes from Pythagoras: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- A quadratic graphs as a **parabola**. Its **vertex** is the turning point, and the **roots** are the x -values where it meets the x -axis, the solutions of $y = 0$. A repeated root touches the axis without crossing; some quadratics have no real roots.
- To solve two equations **graphically**, draw both and read the coordinates of the intersection. The answer is only as accurate as the drawing, which is why an algebraic check matters.

Worked example. a rising line through two given points



Known: $A = (1, 2)$ and $B = (4, 8)$. Use gradient because the line is nonvertical.

$$m = \frac{y_B - y_A}{x_B - x_A} = \frac{8 - 2}{4 - 1} = 2$$

The line equation is $y = mx + c$. Substitute A to find the intercept.

$$c = y_A - mx_A = 2 - 2(1) = 0$$

Thus $y = 2x$. The diagram's horizontal change is 3 and vertical change is 6.

Continue with practice sheet **1.3**. Solve before opening the solutions, and check both the method and the final units.

Geometry: plane, solid and Euclidean geometry

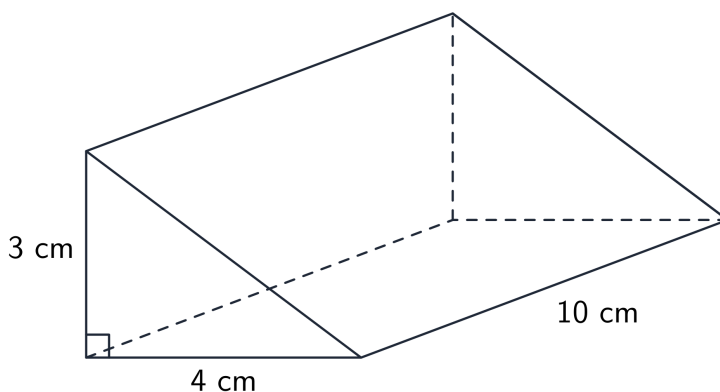
Plane geometry is about flat shapes, solid geometry about three-dimensional ones, and Euclidean geometry is the reasoning that connects them.

Angles on a straight line add to 180° , angles around a point to 360° , and the interior angles of a plane Euclidean triangle to 180° . In a simple **polygon** of n sides the interior angles add to $(n - 2) \times 180^\circ$.

- **Congruent** shapes are identical in size and shape; **similar** shapes have the same shape with all lengths in one ratio.

- **Pythagoras' theorem** holds in a right-angled triangle: $a^2 + b^2 = c^2$, where c is the **hypotenuse**.
- **Area** is measured in square units and **volume** in cubic units. A cylinder has volume $\pi r^2 h$; a sphere has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$.
- $\triangle$ Scaling is not linear. If every length of a solid is doubled, its area is multiplied by $2^2 = 4$ and its volume by $2^3 = 8$.

Worked example. a triangular prism has a constant cross-section



not to scale

Known: perpendicular triangle sides are 3 and 4 cm; prism length is 10 cm. Find the cross-sectional area, then multiply by the prism length.

$$A = \frac{1}{2}bh = \frac{1}{2}(3)(4) = 6 \text{ cm}^2$$

$$V = AL = 6(10) = 60 \text{ cm}^3$$

The sloping triangle side is not its perpendicular height. Volume is not the area of the triangular end.

Continue with practice sheet **1.4**. Solve before opening the solutions, and check both the method and the final units.

Trigonometry

Trigonometry connects the angles of a triangle to its sides.

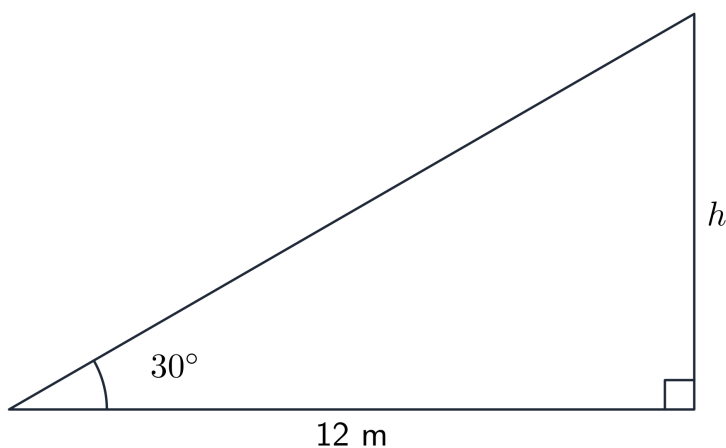
In a right-angled triangle, with θ one of the acute angles, the three ratios are $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ and $\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$.

For a nondegenerate plane Euclidean triangle, the **sine rule** and the **cosine rule** apply:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \quad a^2 = b^2 + c^2 - 2bc \cos A$$

- An angle can be measured in **degrees** or in **radians**, where π radians is 180° . Check which mode your calculator is in before every question.
- The **unit circle** extends the ratios beyond 90° and explains why sin and cos repeat every 360° : they are **periodic**.
- A **bearing** is measured clockwise from north and always written with three figures, such as 075° .
- **Angles of elevation and depression** are measured from the horizontal, upwards and downwards respectively. Add the observer or instrument height if the answer needs height above the ground. With side-side-angle data, the sine rule may allow two triangles: check both supplementary angles against the remaining angle sum.

Worked example. height from a horizontal distance and angle



not to scale

Known: horizontal distance $d = 12$ m and elevation $\theta = 30^\circ$. Height h is opposite and d is adjacent, so use tangent in degree mode.

$$\tan \theta = \frac{h}{d}$$

$$h = d \tan \theta = 12 \tan 30^\circ \approx 6.93 \text{ m}$$

This is the height above the observer's horizontal sight level, not automatically the height above the ground.

Continue with practice sheet **1.5**. Solve before opening the solutions, and check both the method and the final units.

Exponential and logarithmic functions

These two functions describe growth and decay, and they undo each other.

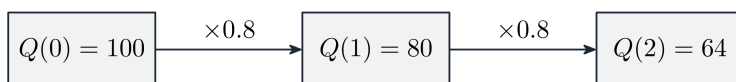
An **exponential function** has the form $y = a^x$ with $a > 0$ and $a \neq 1$: the variable is in the **exponent**. Each unit step multiplies its value by a . It grows when $a > 1$ and decays when $0 < a < 1$. Its y-intercept is 1 and its values stay positive; the x-axis is a horizontal asymptote.

A **logarithm** answers the reverse question. $\log_a y = x$ means exactly $a^x = y$, so $\log_{10} 1000 = 3$. For real logarithms the argument must be positive, and the base must be positive and different from 1. The graph $y = \log_a x$ passes through $(1, 0)$ and has vertical asymptote $x = 0$.

$$\log(mn) = \log m + \log n, \quad \log\left(\frac{m}{n}\right) = \log m - \log n, \quad \log(m^k) = k \log m$$

- For positive arguments and the same valid base, the laws turn multiplication into addition, which is what makes a logarithm useful for solving an equation with the unknown in the exponent.
- **Natural logarithms** use the base $e \approx 2.718$ and are written $\ln$.
- **Compound interest** is exponential: $A = P(1 + r)^n$ after n periods.
- **Exponential decay** has a base between 0 and 1, and can describe a retained proportion each period under stated assumptions. It does not guarantee that a real process follows that model indefinitely.

Worked example. repeated reductions multiply rather than subtract a fixed amount



equal time periods, changing absolute reductions

Known: an invented quantity starts at 100 units and decreases by 20% per whole period. The retained proportion is 0.8, so use a multiplicative model.

$$Q(n) = Q_0(1 - r)^n$$

$$Q(2) = 100(1 - 0.20)^2 = 64$$

The first decrease is 20 units and the second is 16. A fixed subtraction of 20 each period would be a different model.

Continue with practice sheet **1.6**. Solve before opening the solutions, and check both the method and the final units.

GAC010 Mathematics II: Probability, Statistics and Finance

GAC Mathematics

What this module is, and how it is marked

A cheaper headline rate need not produce the cheaper total, and a larger survey need not produce a more representative result. Mathematics II uses explicit rules, calculations and evidence limits to examine such questions.

GAC010 covers sequences, finance, probability and statistics. Your centre's current brief determines assessment tasks, weights, tools and deadlines. A **test**, **projects**, an **examination** and **coursework** may use different instructions; these original practice sheets do not establish their official pattern or university credit.

Show the known data, why the formula applies, the symbolic equation and the substitution. State the unit and interpret the result. A statistical summary describes a defined set of observations or a stated model; it does not automatically establish a cause or a whole-population finding.

All financial amounts, rates, fees and pay rules in the examples are invented teaching conditions. Use their stated timing and rounding assumptions rather than assuming an actual tax or lending rule.

Sequences and series

A **sequence** is an ordered list of numbers; a **series** is their sum.

An **arithmetic sequence** adds a fixed **common difference** d each step. A **geometric sequence** multiplies by a fixed **common ratio** r .

$$u_n = a + (n - 1)d \quad \text{and} \quad u_n = ar^{n-1}$$

- The sums are $S_n = \frac{n}{2}(2a + (n - 1)d)$ and $S_n = \frac{a(1 - r^n)}{1 - r}$ for $r \neq 1$.
- A geometric series with $|r| < 1$ has a **sum to infinity** $S_\infty = \frac{a}{1 - r}$.
- Keep a term separate from a total. Some financial models compound geometrically, while others use simple interest or cash flows. Match the model to the specified rule.

Worked example. row counts differ by a constant amount

row 1: 4 • • • •

row 2: 7 • • • • • • •

row 3: 10 • • • • • • • • • •

Known: row counts are 4, 7 and 10, continuing with the same difference. Thus $a = 4$ and $d = 3$. For the fifth row:

$$u_n = a + (n - 1)d$$

$$u_5 = 4 + (5 - 1)(3) = 16$$

For the first five rows together, use the arithmetic sum, not the fifth term.

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$S_5 = \frac{5}{2}[2(4) + (5 - 1)(3)] = 50$$

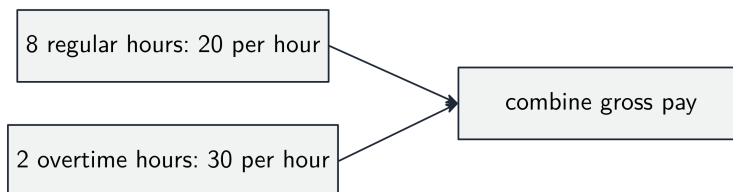
Practice sheet **2.1** gives a progressively harder set with worked solutions.

Finance: earning money and purchasing goods

This unit is the arithmetic of an ordinary life, in English.

- **Gross pay** is what you earn; **net pay** is what arrives after **deductions** such as tax and insurance.
- **Simple interest** is $I = Prt$ — interest on the original amount only.
- **Compound interest** is $A = P(1 + r)^n$ — interest on the interest, which is why it is geometric.
- **Depreciation** is a reduction in recorded value. A fixed-percentage model uses $V_n = V_0(1 - r)^n$; other depreciation conventions can use different rules.
- A **discount** of 20% multiplies by 0.8. Two successive discounts of 20% and 10% multiply by $0.8 \times 0.9 = 0.72$, which is a 28% reduction, not 30%.

Worked example. calculate regular and overtime hours separately



Known: 8 hours at 20 units per hour and 2 overtime hours at 1.5 times that rate. Separate the two categories before adding pay.

$$G = h_r p + h_o k p$$

$$G = 8(20) + 2(1.5)(20) = 220 \text{ units}$$

If the invented fixed deduction is 15 units, take it from gross pay once.

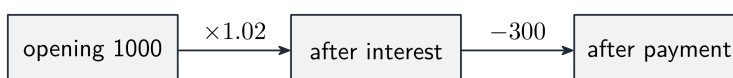
$$N = G - D = 220 - 15 = 205 \text{ units}$$

Practice sheet **2.2** gives a progressively harder set with worked solutions.

Finance: investing money and loans

- Under a stated constant discount rate and timing assumption, the **present value** of a future amount is: $PV = \frac{FV}{(1+r)^n}$.
- An **annuity** has specified payments at regular intervals. A **loan repayment** schedule may instead have changing amounts or an adjusted final payment; specify whether interest comes before or after payment.
- The label **APR** is an annual rate measure whose calculation and included charges depend on the stated convention. This sheet uses explicitly defined period rates, not an unstated APR convention.
- $\triangle$ A nominal 12% teaching rate with monthly rate 1% has effective annual growth $(1.01)^{12} - 1 \approx 12.68\%$. A different compounding frequency or fee schedule needs a different calculation.

Worked example. interest then payment, in that order



Known: opening balance 1000 units, period rate 2%, end-of-period payment 300 units. Interest comes before payment under this stated convention.

$$B_1 = B_0(1 + r) - M$$

$$B_1 = 1000(1.02) - 300 = 720$$

The next period's interest uses 720, not the original principal.

$$B_2 = B_1(1 + r) - M = 720(1.02) - 300 = 434.40$$

Practice sheet **2.3** gives a progressively harder set with worked solutions.

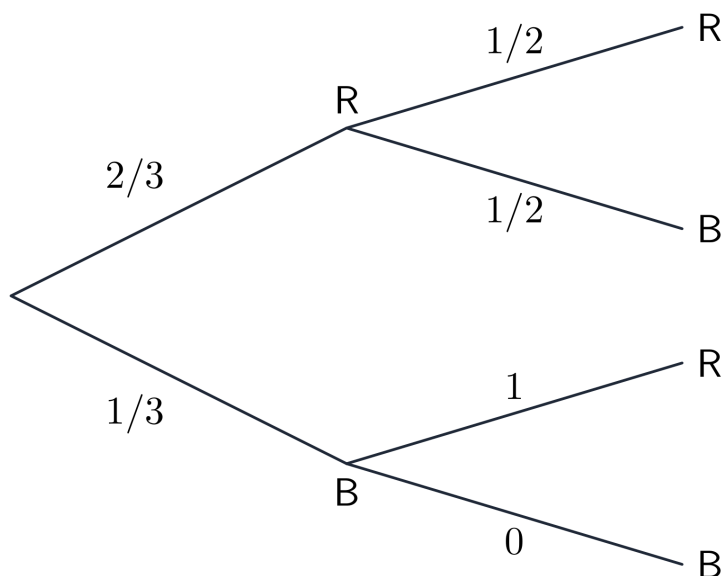
Probability: key concepts

Probability measures how likely an event is, on a scale from 0 to 1.

- For equally likely outcomes, $P(A) = \frac{\text{favourable outcomes}}{\text{total outcomes}}$.
- Events are **mutually exclusive** when they cannot both happen: $P(A \text{ or } B) = P(A) + P(B)$.
- Events are **independent** when one does not affect the other: $P(A \text{ and } B) = P(A) \times P(B)$.
- The **complement** rule, $P(\text{not } A) = 1 - P(A)$, is often the fastest route to an answer containing “at least one”.
- **Counting techniques** supply the denominator: a **permutation** counts ordered selections, a **combination** counts unordered ones.

Conditional probability is $P(A \mid B) = P(A \cap B)/P(B)$ for $P(B) > 0$. A union subtracts its overlap: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Association in a supplied table does not by itself establish a cause.

Worked example. a second draw depends on the first



Known: two red and one blue counter, two random draws without replacement. To get two reds, follow the red-red path and multiply its conditional probabilities.

$$P(RR) = P(R_1)P(R_2 | R_1)$$

$$P(RR) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$

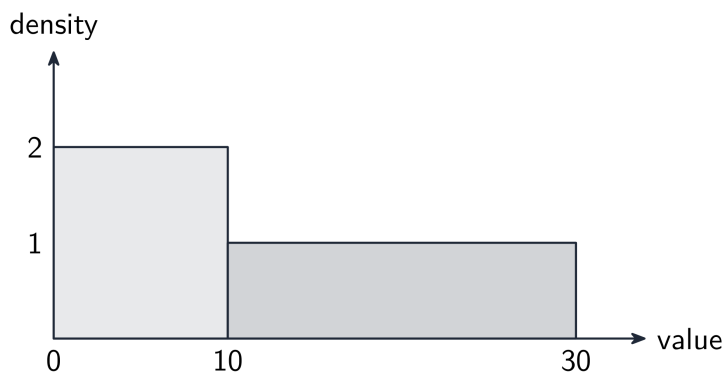
A first blue leaves only red counters; the blue-blue path has probability zero. The draws are not independent.

Practice sheet **2.4** gives a progressively harder set with worked solutions.

Statistics: collecting and displaying data

- A **population** is everyone you want to describe; a **sample** is who you actually measure.
- Generalising to a population needs a suitably **representative** sample and an honest account of selection and nonresponse. A class survey can describe that class without establishing a school-wide percentage.
- **Qualitative data** are categories; **quantitative data** measure amounts, either **discrete** (counted) or **continuous** (measured).
- **Bias** enters through who you ask, who answers, and how the question is worded, and arithmetic alone does not repair missing or systematically selected observations.

Worked example. equal bar heights can conceal unequal frequencies



Known: class 0 to 10 has frequency 20; class 10 to 30 has frequency 20. Widths differ, so calculate frequency density.

$$D = \frac{f}{w}$$

$$D_1 = \frac{20}{10} = 2$$

$$D_2 = \frac{20}{20} = 1$$

The second bar is half as high but twice as wide. Both areas represent 20 observations.

Practice sheet **2.5** gives a progressively harder set with worked solutions.

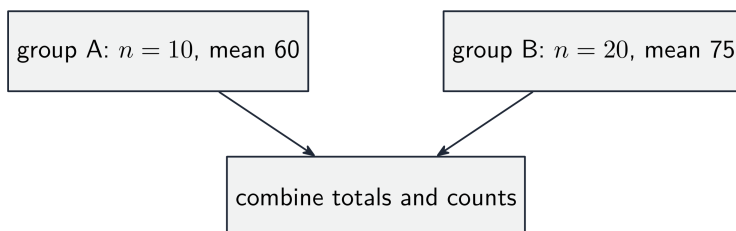
For unequal-width histogram classes, height is frequency divided by class width so that bar area represents frequency. Keep categories in separated bars and numeric continuous intervals in a histogram.

Statistics: measures of central location

- The **mean** is the total divided by the count. It uses every value, and every outlier.
- The **median** is the middle value in order. Moving an extreme value without changing the ordered middle may leave it unchanged.
- The **mode** is the most frequent value, and it can identify the most frequent category without imposing a numerical scale. A distribution may have more than one mode or no unique mode.

- $\triangle$ Choose the summary for the question. A high extreme can pull the mean above the median. The mean still measures total per observation; the median locates the ordered middle. Neither says that most observations equal it.

Worked example. group size determines a combined mean



Known: group A has 10 scores with mean 60; group B has 20 with mean 75. Recover each group's total before combining.

$$\bar{x} = \frac{n_A \bar{x}_A + n_B \bar{x}_B}{n_A + n_B}$$

$$\bar{x} = \frac{10(60) + 20(75)}{10 + 20} = 70$$

The unweighted mean of 60 and 75 would be 67.5, which ignores the unequal counts.

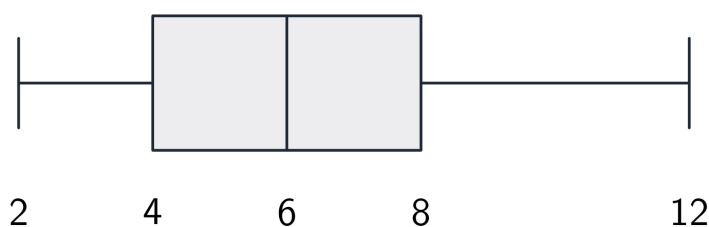
Practice sheet **2.6** gives a progressively harder set with worked solutions.

Statistics: measures of variability

- The **range** is the largest minus the smallest — the full span, so an extreme can change it substantially.
- The **interquartile range (IQR)** is the difference $Q_3 - Q_1$ spanning the central half by the stated quartile convention. A **box plot** shows quartiles and a median, with whisker rules that must be specified.
- The **standard deviation** summarises spread using squared deviations. For a complete population, use: $\sigma = \sqrt{\sum(x - \mu)^2/n}$. A sample estimate uses $s = \sqrt{\sum(x - \bar{x})^2/(n - 1)}$ under that stated convention. Variance has squared units; standard deviation has the original units.
- A larger standard deviation means a wider spread. Two data sets can share a mean and describe completely different situations.

Worked example. read a five-number summary rather than every observation

whiskers: minimum to maximum



Known five-number summary: minimum 2, $Q_1 = 4$, median 6, $Q_3 = 8$, maximum 12. The overall span and middle-half span are different.

$$R = x_{max} - x_{min} = 12 - 2 = 10$$

$$IQR = Q_3 - Q_1 = 8 - 4 = 4$$

The box does not identify every original value, the sample size or why the spread arose.

Practice sheet **2.7** gives a progressively harder set with worked solutions.

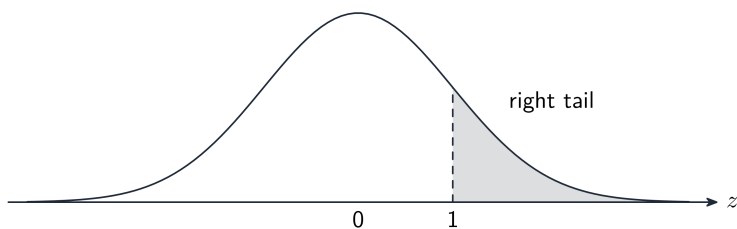
For sheet 2.7, quartiles are medians of ordered half-lists, excluding the overall median for an odd count. Whiskers run to the minimum and maximum. Do not mix that convention with software interpolated quartiles or outlier-fence whiskers.

Distributions: the normal distribution

- The **normal distribution** is a continuous symmetric bell-shaped model specified by its mean and positive standard deviation. Those two summaries alone do not establish that observed data follow this model.
- Under a normal model, the approximate empirical rule is: about 68% of values lie within one standard deviation of the mean, 95% within two, and 99.7% within three.
- A **z-score** says how many standard deviations a value sits from the mean: $z = \frac{x - \mu}{\sigma}$, which is what makes two different scales comparable.

For a normal model, use cumulative area $\Phi(z) = P(Z \leq z)$ and symmetry $\Phi(-z) = 1 - \Phi(z)$. An interval subtracts its two cumulative bounds. Expected count is sample size times probability and need not be an integer; an observed count is an integer.

Worked example. a threshold is an area beyond a standardised value



Known: an invented measurement model has $\mu = 50$ and $\sigma = 10$. For values above 60, standardise the threshold.

$$z = \frac{x - \mu}{\sigma} = \frac{60 - 50}{10} = 1$$

Use the complement of the supplied cumulative area.

$$P(X > 60) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$$

This is a model probability, not a guarantee that exactly 15.87% of a finite sample exceeds 60.

Practice sheet **2.8** gives a progressively harder set with worked solutions.

GAC016 Mathematics III: Calculus & Advanced Applications

GAC Mathematics

What this module is, and how it is marked

An average speed does not tell you every instantaneous speed, and a signed displacement does not always equal total distance. Calculus makes those distinctions precise.

GAC016 develops **differentiation**, **integration** and their applications. Your centre's current brief determines assessment tasks and grading requirements; these original practice sheets do not establish a university credit decision or an official examination pattern.

State the independent variable, domain and units. The notation dy/dx is a derivative with respect to x ; a dot over y normally denotes a derivative with respect to time. Those notations describe the same operation only when their independent variables agree.

For applications, describe what a rate or accumulated value means in the stated model. A negative volume-change rate of 3 cubic centimetres per second means volume is falling at that rate, rather than that the volume itself is negative.

Differentiation

A **derivative** is a rate of change: how fast y changes as x changes. Geometrically it is the **gradient of the tangent** at a point.

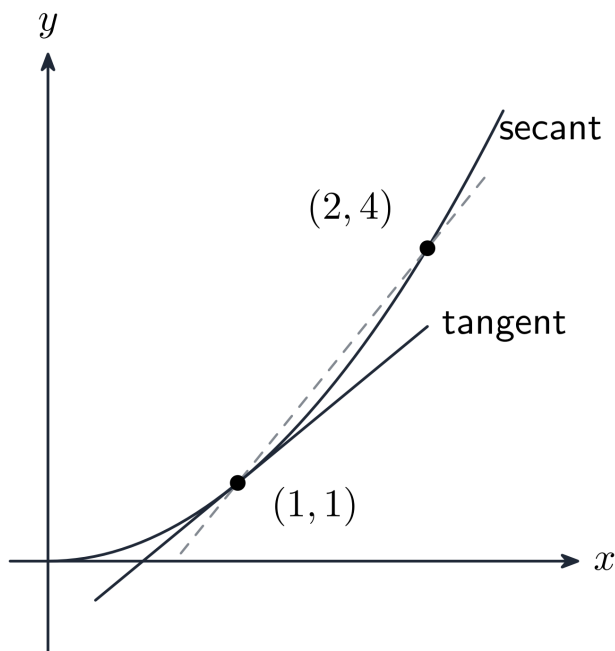
$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

That limit is the definition. **Differentiation from first principles** uses the difference quotient for nonzero h and then takes its limit. Unequal one-sided limits, as for $|x|$ at zero, prevent a derivative even when the function is continuous.

- The **power rule**, on a domain where the power is differentiable: if $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$.
- The **product rule**: $(uv)' = u'v + uv'$.
- The **quotient rule**, where $v \neq 0$: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$.
- The **chain rule**: if $y = f(g(x))$ then $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$.

- **Stationary points** occur where $f'(x) = 0$. The **second derivative** then classifies a stationary point when it is nonzero: negative gives a local **maximum** and positive a local **minimum**. A zero second derivative is inconclusive; use derivative signs or other evidence. A stationary inflection can be neither an extremum.

Worked example. a tangent slope differs from a secant slope



Known: $f(x) = x^2$. The secant between x equal to 1 and 2 has slope 3, but the derivative at x equal to 1 is 2.

$$m_{sec} = \frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{2 - 1} = 3$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x$$

$$m_{tan} = f'(1) = 2(1) = 2$$

The average slope over a finite interval need not equal the instantaneous slope at either endpoint.

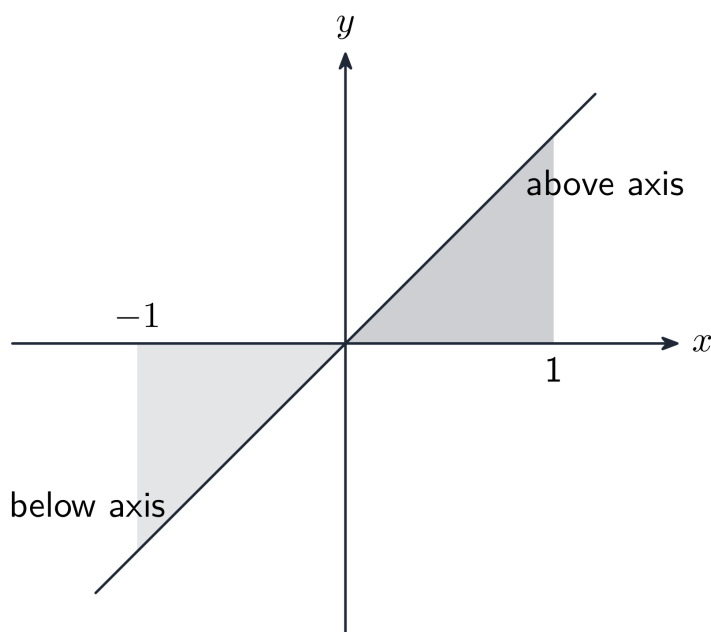
Use original practice sheet **3.1** to test the method, domain and interpretation against its solutions.

Integration

Indefinite integration finds antiderivatives. Definite integration measures signed accumulation, which can differ from geometric area. For a continuous integrand, the **fundamental theorem of calculus** connects a definite integral to the difference of antiderivative values at its limits.

- An **indefinite integral** has no limits and needs the **constant of integration** : $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$.
- A **definite integral** has limits and gives a number: $\int_a^b f(x) dx = F(b) - F(a)$.
- **Integration by substitution** reverses the chain rule.
- $\triangle$ Forgetting $+c$ on an indefinite integral is the single most frequent lost mark in the module, and it is lost on questions you have otherwise answered correctly.

Worked example. negative and positive areas can cancel



Known: $f(x) = x$ from negative 1 to 1. Its definite integral is zero because the negative and positive contributions cancel.

$$I = \int_{-1}^1 x dx = \left[\frac{x^2}{2} \right]_{-1}^1 = \frac{1}{2} - \frac{1}{2} = 0$$

Geometric area instead adds the magnitudes of the two triangular regions.

$$A = - \int_{-1}^0 x \, dx + \int_0^1 x \, dx = \frac{1}{2} + \frac{1}{2} = 1$$

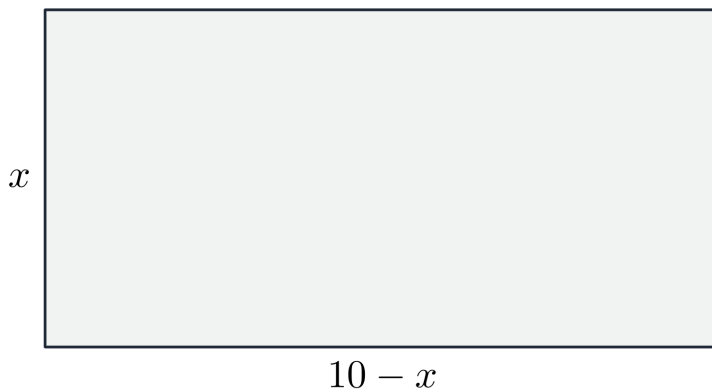
Use original practice sheet **3.2** to test the method, domain and interpretation against its solutions.

Advanced applications

- **Optimisation** finds the largest or smallest value of a quantity: write the quantity as a function of one variable on a stated feasible domain, differentiate, find candidates and justify the required optimum. Check included boundaries and integer constraints where relevant.
- **Rates of change** chain together: $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$.
- **Area between two curves** is $\int_a^b (f(x) - g(x)) \, dx$ where f is the upper curve on that whole interval. Split at crossings if their order changes.
- **Marginal cost and revenue** are derivatives of stated continuous cost and revenue models. They are instantaneous rates, not automatically exact discrete-unit changes.

Worked example. a fixed perimeter leaves one area variable

perimeter 20 units



not to scale

Known: a rectangle has perimeter 20, so length plus width is 10. Let width be x ; length is $10 - x$ and $0 < x < 10$.

$$A(x) = x(10 - x)$$

$$A'(x) = 10 - 2x$$

The stationary candidate is $x = 5$. Since $A''(x) = -2 < 0$ and the area approaches zero at both domain ends, this candidate is the global maximum. It is a 5 by 5 square with area 25.

Use original practice sheet **3.3** to test the method, domain and interpretation against its solutions.

GAC024 Discrete Mathematics

GAC Mathematics

What this module is, and how it is marked

A repeated set member is counted once, a binary carry may exceed a fixed width, and the fewest-edge route may not have the smallest weight. Discrete mathematics makes those rules explicit.

GAC024 covers sets, counting systems, binary logic, algorithms and networks. Your centre's current brief determines assessment tasks, tools, weights and deadlines. These original practice sheets do not establish official marking rules or a university credit decision.

State the universe, representation width, allowed inputs or graph assumptions before solving. Show enough working for another reader to reproduce the result and distinguish a mathematical model from its real implementation.

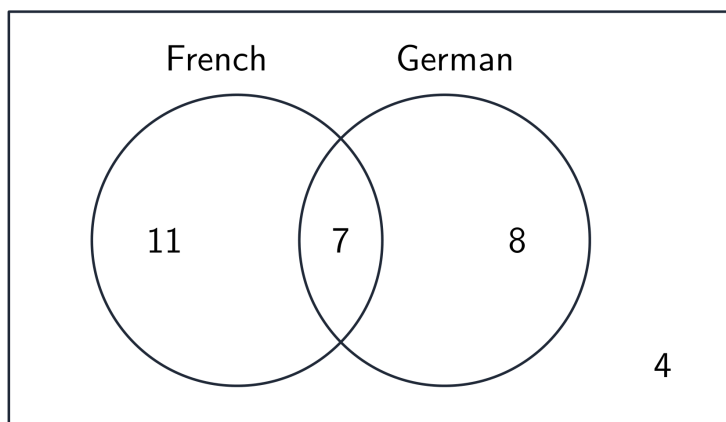
Sets, relations and functions

- A **set** is a collection of distinct objects. Order and repetition do not matter.
- **Union** $A \cup B$ is everything in either; **intersection** $A \cap B$ is what is in both; the **set complement** is everything in the stated universe but outside the set.
- A **subset** has all its elements inside another set.
- A **relation** pairs elements of two sets. A **function** is a relation where each input in its stated domain has exactly one output. Different inputs may share an output; an inverse relation is a function only when outputs uniquely identify their inputs.
- A **Venn diagram** turns a set problem into a picture, and can show the disjoint regions and their counts. Check that those regions add to the supplied universe total.

The **inclusion-exclusion principle** subtracts the twice-counted overlap once: $|A \cup B| = |A| + |B| - |A \cap B|$.

Worked example. subtract an overlap only once

universe: 30 learners



Known: 30 learners, 18 study French, 15 German, and 7 both. The overlap is included in both subject totals.

$$|F \cup G| = |F| + |G| - |F \cap G| = 18 + 15 - 7 = 26$$

$$N_{\text{neither}} = |U| - |F \cup G| = 30 - 26 = 4$$

French-only is $18 - 7 = 11$ and German-only $15 - 7 = 8$. The four disjoint regions sum to 30.

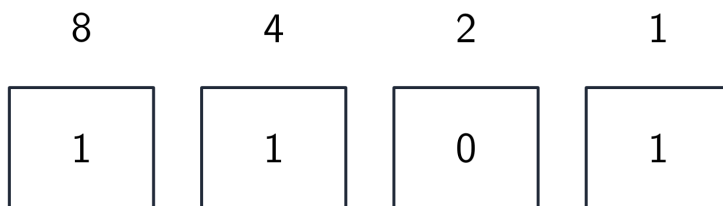
Practice sheet 4.1 includes progressively harder problems and independently checked solutions.

Counting systems

- A positional **number base** b uses digits from zero to b minus one and place weights b^i . **Decimal** uses ten, **binary** two, **hexadecimal** sixteen.
- Every digit's value is its **place value** : in binary the places are 1, 2, 4, 8, 16 and so on.
- Hexadecimal is shorthand for binary: one hex digit is exactly four bits, so conversion can group a stated-width binary pattern into four-bit blocks. Leading zeros preserve width while leaving the unsigned value unchanged.

For n unsigned bits, values run from zero to $2^n - 1$. Distinguish an unrestricted sum from a stored fixed-width result; a wraparound rule, if explicitly given, keeps the low n bits.

Worked example. place weights determine the decimal value



place weights above binary digits

Known numeral 1101_2 . Use weights from right to left: 1, 2, 4 and 8.

$$V = \sum d_i 2^i$$

$$V = 1(8) + 1(4) + 0(2) + 1(1) = 13$$

The same value is D in hexadecimal. Leading zeros would not change this nonnegative value but can record an intended width.

Practice sheet **4.2** includes progressively harder problems and independently checked solutions.

Binary applications

- **Binary arithmetic** adds like decimal, carrying at 2 instead of at 10.
- A **bit** is one binary digit; a **byte** is eight.
- **Boolean algebra** works on true and false with AND, OR and NOT.
- A **truth table** lists every Boolean input combination and output. Matching every row proves equivalence for the same finite Boolean inputs; it does not prove physical circuit timing or real-system security.
- **Logic gates** implement stated operations, and a **logic circuit** connects them. Trace the abstract logic according to its connections and input conventions.

Use inclusive OR and explicit brackets. De Morgan gives $\neg(A \wedge B) = (\neg A) \vee (\neg B)$. Bitwise NOT inverts only the stated width, not an unspecified infinite representation.

Worked example. an OR output is inverted by NOT



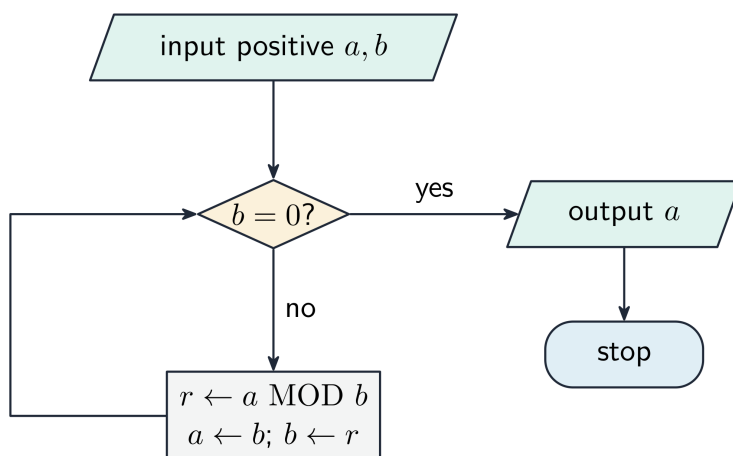
Known: $Y = \neg(A \vee B)$. Inclusive OR is false only when both inputs are false; NOT reverses that result. In row order $(A, B) = (0, 0), (0, 1), (1, 0), (1, 1)$, the output column is 1, 0, 0, 0. De Morgan gives equivalent expression $(\neg A) \wedge (\neg B)$.

Practice sheet 4.3 includes progressively harder problems and independently checked solutions.

Algorithms

- An **algorithm** describes unambiguous steps for a task. A procedure solving the stated finite task must terminate and give the required result for its allowed inputs.
- A **flowchart** draws it: a decision is a diamond, a process a rectangle.
- **Pseudocode** represents its steps without requiring a particular implementation language. State assignment, loop bounds and index conventions before tracing.
- **Tracing** an algorithm — a table with one column per variable and one row per step — records its actual updates. A trace checks the chosen input; a claim for all allowed inputs also needs a correctness argument.
- **Efficiency** matters: a linear search can stop early but may inspect all n items. Binary search repeatedly discards half of an ordered search range; its logarithmic comparison count requires the sorted-data and bound conventions.

Worked example. repeat a remainder step until the second number is zero



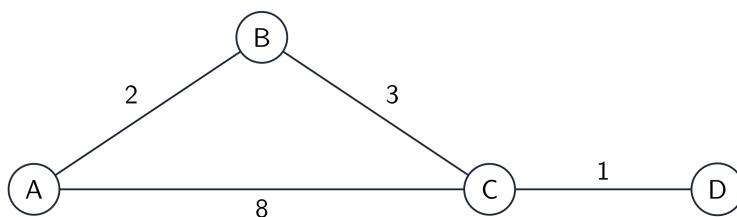
Known: start with positive integers a equal to 10 and b equal to 6. While b is nonzero, compute r as $a \text{ MOD } b$, then set a to b and b to r . Pairs after complete iterations are $(6,4)$, $(4,2)$, $(2,0)$, giving output 2. Temporary r preserves the remainder before a and b change. Each nonzero remainder is smaller than the previous positive b , supporting termination.

Practice sheet 4.4 includes progressively harder problems and independently checked solutions.

Graphs and networks

- A **graph** is a set of **vertices** joined by **edges**. It models anything with connections: roads, friendships, dependencies.
- For a simple undirected graph with no loops or repeated edges, the **degree** counts incident edges. Every edge contributes two to the total degree sum.
- A **tree** is a connected graph with no cycles, and a finite tree with n vertices has n minus 1 edges. Some hierarchical models use trees, but actual systems can also contain cross-links or cycles.
- A **shortest path** problem asks for the cheapest route between two vertices, by total weight under the stated constraints, rather than by the number of edges alone. A minimum spanning tree instead connects every vertex without cycles and minimises total included edge weight.

Worked example. compare total route weight, not the number of edges



Known edge weights are $AB = 2$, $BC = 3$, $AC = 8$ and $CD = 1$. The path A-C-D has weight 9, while A-B-C-D has weight 6. Therefore the three-edge path is shorter by weight despite having more edges. The minimum spanning tree for this small network uses AB, BC and CD with total 6; the agreement of totals here does not make the tasks identical.

Practice sheet 4.5 includes progressively harder problems and independently checked solutions.