Mechanics 3

Edexcel IAL M3 (WME03/01); elastic strings, SHM, circular motion and centres of mass

Formulae to memorise

Family	Must know
Hooke's law	$T = \frac{\lambda e}{L}$
elastic energy	$\frac{\lambda e^2}{2L}$
SHM	$\ddot{x} = -\omega^2 x$, **period** $\frac{2\pi}{\omega}$, $v^2 = \omega^2(a^2 - x^2)$
circles	$r\omega^2 = \frac{v^2}{r}$ towards the centre
solid COM	$\overline{x} = \frac{\int xy^2\,dx}{\int y^2\,dx}$
a = f(x)	$a = v\frac{dv}{dx}$

Worked: acceleration as a function of x

1. **v dv/dx** $\frac{1}{2}v^2 = 2x^2 - 2x + \frac{1}{2}$
2. **Square** $v^2 = (2x-1)^2$: $v = 1 - 2x$
3. **Then in t** $\frac{dv}{dt} = -2v$: $\ln v = -2t + C$

Decision

$v = e^{2-2t}$

Hooke's law and elastic energy

Tension

$T = \frac{\lambda e}{L}$

proportional to extension

Elastic energy

$\frac{\lambda e^2}{2L}$

joins kinetic and gravitational

A string with a midpoint

Elastic energy gained: $\frac{1}{2}mga$

- $\lambda = mg$, **natural length** $3a$.
- Treat each half separately.

Worked: three energy stores

1. **Gravity gives** $mg \cdot \frac{3a}{2} = \frac{3}{2}mga$
2. **It pays for** ΔKE, elastic $\frac{1}{2}mga$, resistance $\frac{1}{5}mg \cdot \frac{3a}{2}$
3. **Solve** $\frac{3}{2}mga = \frac{1}{2}mV^2 - \frac{3}{4}mga + \frac{1}{2}mga + \frac{3}{10}mga$

Decision

$V = \sqrt{\frac{29ag}{10}}$

Worked: a hinged rod on a string

1. **Geometry** B is $1.6a$ across, $1.2a$ below A; midpoint $0.8a$ across
2. **Moments about A** $T \cdot 1.6a = mg \cdot 0.8a$: $T = \frac{1}{2}mg$
3. **Hooke** $\frac{1}{2}mg = \frac{2mg}{L}(1.2a - L)$

Decision

$L = 0.96a = \frac{24}{25}a$

Worked: SHM between two strings

1. **Equilibrium** $AE = 4.5L$: both tensions $\frac{mg}{2}$
2. **Displace by x** $m\ddot{x} = \frac{2mg}{2L}(\frac{L}{2} - x) - \frac{mg}{3L}(\frac{3L}{2} + x)$
3. **Simplify** $\ddot{x} = -\frac{4g}{3L}x$

Decision

SHM, **period** $\pi\sqrt{\frac{3L}{g}}$

The slow third

Time with $|v| \leq \frac{v_{max}}{2}$: $\pi\sqrt{\frac{L}{3g}}$

- **Amplitude** $\frac{L}{2}$, $v_{max} = \sqrt{\frac{gL}{3}}$.
- Time fractions come from the sine.

A circle on a smooth table

$T = \frac{mg}{2}$, $R = \frac{3mg}{4}$

- Resolve towards the centre.
- Resolve vertically.

Leaving a sphere

$$\tan\alpha = \frac{2}{3}$$

- Energy between levels.
- Leaves when R $= 0$.

Worked: after leaving

1. **Leaving speed** $v^2 = 2gr(\cos\alpha - \sin\alpha)$, horizontal part $v\sin\alpha$
2. **Level with O** $\frac{1}{2}mW^2 = mgr\cos\alpha$
3. **Direction** $\cos^2\theta = \frac{v^2\sin^2\alpha}{W^2} = \frac{4}{39}$

Decision

$\cos\theta = \frac{2}{\sqrt{39}}$

The hemisphere by discs

$\bar{x} = \frac{3a}{8}$ from the plane face

- Weight discs by $\pi y^2\,dx$.
- Derivable, not just quoted.

Worked: a carved hemisphere

1. **Masses** $8 : 1 : 7$
2. **Along the axis** $8 \cdot \frac{3a}{8} - 1 \cdot \frac{3a}{16} = 7\bar{y}$: $\bar{y} = \frac{45a}{112}$
3. **Across** $-1 \cdot \frac{a}{2} = 7\bar{x}$: $|\bar{x}| = \frac{a}{14}$

Decision

Tilted: $\tan\theta = \frac{a/14}{45a/112} = \frac{8}{45}$

Check yourself

Question	Answer
$a = f(x)$: write a as?	$v\frac{dv}{dx}$
Elastic energy?	$\frac{\lambda e^2}{2L}$
What proves SHM?	$\ddot{x} = -\omega^2 x$
Leaving a surface?	$R = 0$
Hemisphere COM?	$\frac{3a}{8}$ from the face