Further Pure Mathematics 3

Edexcel IAL FP3 (WFM03/01); hyperbolic functions, conics, calculus, vectors and matrices

Unit FP3 at a glance

Topic	**Core tool**
hyperbolic functions	definitions in e^x; log forms of the inverses
conics	**eccentricity**: $PS = e \cdot PN$
calculus	reduction formulae, **arc length**, surfaces
vectors	$\mathbf{a} \times \mathbf{b}$, planes, distances
matrices	3×3 inverse, eigenvalues

Built from exponentials

$\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$

- $\cosh^2 x - \sinh^2 x \equiv 1$
- $\cosh 2x \equiv \cosh^2 x + \sinh^2 x$

Worked: an identity from the definitions

1. **Expand** $\frac{1}{2}(e^{6x}+e^{-6x})+\frac{1}{2}(e^{4x}+e^{-4x})$
2. **Pair** $=\frac{1}{2}(e^{5x}+e^{-5x})(e^{x}+e^{-x})=2\cosh 5x\cosh x$
3. **Solve** $\cosh 6x+\cosh 4x=8\cosh x$: $\cosh 5x=4$

Decision

$x=\pm\frac{1}{5}\ln(4+\sqrt{15})$

Worked: a quadratic in e to the x

1. **Definitions** $(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = 2$
2. **Times 2e^x** $3e^{2x} - 4e^x + 1 = 0$
3. **Factor** $(3e^x - 1)(e^x - 1) = 0$

Decision

$x = 0$ or $x = -\ln 3$

The inverses as logarithms

Inverse	**Logarithmic form**
$\operatorname{arsinh} x$	$\ln(x+\sqrt{x^2+1})$
$\operatorname{arcosh} x$	$\ln(x+\sqrt{x^2-1})$, $x \geq 1$
$\operatorname{artanh} x$	$\frac{1}{2}\ln\frac{1+x}{1-x}$, $\lvert x\rvert < 1$

Worked: deriving arsinh

1. **Set** $w = \sinh x = \frac{1}{2}(e^x - e^{-x})$
2. **Quadratic** $e^{2x} - 2we^x - 1 = 0$
3. **Choose** $e^x = w + \sqrt{w^2 + 1}$, since $e^x > 0$

Decision

$$\operatorname{arsinh} w = \ln(w + \sqrt{w^2 + 1})$$

Ellipse and hyperbola

	Ellipse	**Hyperbola**
curve	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
parameters	$(a\cos t,\ b\sin t)$	$(a\sec t,\ b\tan t)$
eccentricity	$b^2 = a^2(1 - e^2)$	$b^2 = a^2(e^2 - 1)$

Worked: the hyperbola's foci

$e = \frac{\sqrt{113}}{8}$, foci $(\pm\sqrt{113}, 0)$

- H: $\frac{x^2}{64} - \frac{y^2}{49} = 1$
- Asymptotes $y = \pm\frac{b}{a}x$.

Worked: a circle through both foci

1. **Gradient** $\frac{dy}{dx} = \frac{7\sec\theta}{8\tan\theta}$
2. **y-intercepts** tangent: $Q(0, -7\cot\theta)$; normal: $R(0, \frac{113}{7}\tan\theta)$
3. **Circle on QR** meets the x-axis where $x^2 = -qr = 113$

Decision

Both foci $(\pm\sqrt{113}, 0)$ lie on it, for every θ.

When a line is a tangent

Ellipse

$c^2 = a^2m^2 + b^2$

any gradient works

Hyperbola

$c^2 = a^2m^2 - b^2$

needs $|m| > \frac{b}{a}$

Six derivatives

Function	Derivative	Function	Derivative
$\sinh x$	$\cosh x$	$\operatorname{arsinh} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh x$	$\sinh x$	$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh x$	$\operatorname{sech}^2 x$	$\operatorname{artanh} x$	$\frac{1}{1-x^2}$

Worked: the domain does real work

1. **Chain rule** $\frac{1}{\sqrt{1+x^2}} + \frac{-1/x^2}{\sqrt{1+1/x^2}}$
2. **Simplify** $= \frac{x-1}{x\sqrt{1+x^2}}$
3. **Stationary** $x = 1$

Decision

$y = 2\operatorname{arsinh} 1 = 2\ln(1+\sqrt{2})$

Worked: complete the square, then by parts

1. **Square** $4x^2 + 8x + 9 = 4\left((x+1)^2 + \frac{5}{4}\right)$
2. **Standard form** $\int \frac{dx}{\sqrt{4x^2+8x+9}} = \frac{1}{2}\operatorname{arsinh}\frac{2x+2}{\sqrt{5}} + c$
3. **By parts** $\int \operatorname{arcosh} 3x\,dx = x\operatorname{arcosh} 3x - \int \frac{3x}{\sqrt{9x^2-1}}dx$

Decision

$x\operatorname{arcosh} 3x - \frac{1}{3}\sqrt{9x^2-1} + c$

Four integrals to know cold

Integrand	Integral
$\frac{1}{\sqrt{x^2+a^2}}$	$\operatorname{arsinh}\frac{x}{a}+c$
$\frac{1}{\sqrt{x^2-a^2}}$	$\operatorname{arcosh}\frac{x}{a}+c$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\arctan\frac{x}{a}+c$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert+c$

Worked: a reduction formula

1. **By parts** $u = x^n$, $v = \frac{2}{3}\sqrt{3x-2}$
2. **Collect** $(3+6n)I_n = 4nI_{n-1} + 8 \cdot 6^n - 2$
3. **Iterate** $I_1 = 6$, $15I_2 = 334$

Decision

$I_3 = \frac{3322}{35}$

Rotate the arch

$$S = 2\pi \int y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

- **Arc length** $s = \int \sqrt{1 + y'^2}\,dx$.
- Calculator banned.

Worked: the surface with a sinh substitution

1. **Substitute** $2\sin 2x = \sinh\theta$: $4\cos 2x\,dx = \cosh\theta\,d\theta$
2. **Reduce** $S = \frac{\pi}{2}\int \cosh^2\theta\,d\theta = \frac{\pi}{4}[\sinh\theta\cosh\theta + \theta]$
3. **Limits** 0 to $\operatorname{arsinh} 2 = \ln(2+\sqrt{5})$

Decision

$S = \frac{\pi}{4}\left(2\sqrt{5} + \ln(2+\sqrt{5})\right)$

What the products measure

Quantity	Meaning
$\mathbf{a} \times \mathbf{b}$	perpendicular to both
$\lvert\mathbf{a} \times \mathbf{b}\rvert$	parallelogram area
$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$	volume; zero means coplanar
$\frac{\lvert\mathbf{x}\cdot\mathbf{n}-p\rvert}{\lvert\mathbf{n}\rvert}$	point-to-plane distance

Worked: planes to a line to a distance

1. **Normal** $(7,5,-6)\times(-3,1,2)=2(8,2,11)$: $\mathbf{r}\cdot(8,2,11)=25$
2. **Line of meeting** with $x-y+z=7$: direction $(13,3,-10)$
3. **Common normal** $(13,3,-10)\times(1,-1,-1)=(-13,3,-16)$

Decision

Shortest distance $\frac{11}{\sqrt{434}}$

Eigenvalues from one eigenvector

Each row of $\mathbf{Av}$ is λ times that component of $\mathbf{v}$.

- $\mathbf{Av} = \lambda\mathbf{v}$, row by row.
- Trace and determinant check for free.

Worked: eigenvalues of M

1. **Row 1** $1 - 2 + 4 = 3$: $\lambda = 3$
2. **Rows 2 and 3** $2a + b = 3$, $a + b = 1$: $a = 2$, $b = -1$
3. **Trace and det** trace 2, $\det = -6$: others sum -1, product -2

Decision

$\lambda = 3,\ 1,\ -2$

Worked: when is A singular?

1. **Expand row 1** $1(10) - k(32) + 2(-23)$
2. **Determinant** $\det \mathbf{A} = -36 - 32k$
3. **Check** $(1, k, 2) \cdot (10, -32, -23) = -36 - 32k$

Decision

Singular when $k = -\frac{9}{8}$

Check yourself

Question	**Answer**
$\frac{d}{dx}\cosh x$?	$\sinh x$
$\operatorname{arsinh} x$ as a log?	$\ln(x+\sqrt{x^2+1})$
Hyperbola: $b^2 =$?	$a^2(e^2-1)$
$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=0$?	coplanar
Eigenvalues: sum and product?	trace, determinant