Further Pure Mathematics 2

Edexcel IAL FP2 (WFM02/01)

Unit FP2 at a glance

Topic	**Core tool**
inequalities	case split, or square both sides
series	the method of differences
complex	$e^{i\theta}$, de Moivre, loci, $w = \frac{az+b}{cz+d}$
differential equations	**integrating factor**; CF + PI
series and polar	Maclaurin, Taylor; $\frac{1}{2}\int r^2\,d\theta$

Two safe moves, one banned

Safe

split: $|f| = f$ or $-f$

square: $|A| > |B| \iff A^2 > B^2$

Banned

multiplying by an expression

whose sign is unknown

Worked: one branch dies

$x < -1$ or $x > 3$

- Solve $|x^2 - 1| > 2(x + 1)$.
- A dead branch is a result.

Worked: a modulus in the denominator

1. **x ≥ 0** $(x-2)(x+4) > 15x$: $x^2 - 13x - 8 > 0$
2. **x < 0** $(x-2)(4-x) > 15x$: $(x+1)(x+8) < 0$
3. **Combine** $x > \frac{13+\sqrt{201}}{2}$; $-8 < x < -1$

Decision

$-8 < x < -1$ or $x > \frac{13+\sqrt{201}}{2}$

The method of differences

1. **Split** $\frac{1}{(r+1)(r+2)} = \frac{1}{r+1} - \frac{1}{r+2}$
2. **Cancel** $(\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{n+1} - \frac{1}{n+2})$
3. **Survivors** $\frac{1}{2} - \frac{1}{n+2} = \frac{n}{2(n+2)}$

Decision

$\frac{n}{2(n+2)} > 0.49$ needs $n > 98$: smallest $n = 99$

Euler's relation and de Moivre

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

- $z = re^{i\theta}$, so $z^n = r^n e^{in\theta}$
- Proved by induction for integer n.

Worked: sin 5θ

1. **Imaginary part** $5c^4s - 10c^2s^3 + s^5$
2. **In sine only** $\sin 5\theta = 16s^5 - 20s^3 + 5s$
3. **5θ =** π $16s^4 - 20s^2 + 5 = 0$: $s^2 = \frac{5 \pm \sqrt{5}}{8}$

Decision

$\sin \frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4}$, the smaller root

Five fifth roots on a circle

$$z = \sqrt{2}\, e^{i\left(-\frac{2\pi}{15} + \frac{2\pi k}{5}\right)},\ k = 0, \ldots, 4$$

- $|w| = \sqrt{32} = 4\sqrt{2}$, $\arg w = -\frac{2\pi}{3}$
- Exact polar form, never decimals.

The loci toolkit

Locus	**Shape**
$\lvert z-a\rvert = b$	circle, centre a, radius b
$\lvert z-a\rvert = \lvert z-b\rvert$	**perpendicular bisector** of a, b
$\lvert z-a\rvert = k\lvert z-b\rvert,\ k \neq 1$	a circle: square and collect
$\arg(z-a) = \beta$	half-line from a

Worked: the image of a circle

$|9i - w| = |w + i|$: the bisector of $9i$ and $-i$

- Make z the subject first.
- Then take moduli.

The integrating factor

1. **Factor** $\frac{dy}{dx} + 2y = e^{-x}$: $\text{IF} = e^{\int 2\,dx} = e^{2x}$
2. **Product** $\frac{d}{dx}(ye^{2x}) = e^{x}$
3. **Integrate** $ye^{2x} = e^{x} + C$

Decision

$y = e^{-x} + Ce^{-2x}$

Worked: substitution, then a factor

1. **Substitute** $\frac{dz}{dx} - z\cot x = \tan x \sec x$
2. **Factor** IF $= \csc x$: $\frac{d}{dx}(z\csc x) = \sec^2 x$
3. **Solve** $z = \sin x \tan x + C\sin x$, $C = \frac{1}{2\sqrt{3}}$

Decision

At $x = \frac{\pi}{3}$: $y^2 = \frac{4}{7}$, $y = \pm\frac{2}{\sqrt{7}}$

The complementary function

Auxiliary roots	**Complementary function**
distinct m_1, m_2	$Ae^{m_1 x} + Be^{m_2 x}$
repeated m	$(A + Bx)e^{mx}$
complex $p \pm qi$	$e^{px}(A\cos qx + B\sin qx)$

Choosing the particular integral

Right-hand side	**Trial**
ke^{px}	λe^{px}; $\lambda x e^{px}$ if p is a root
$a + bx$	$\lambda + \mu x$
$m\cos\omega x + n\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$

Worked: resonance, then conditions

1. **PI** left side $= 13\lambda e^{3x}$, so $\lambda = 6$
2. **CF** $(4m+1)(m-3) = 0$: $Ae^{-x/4} + Be^{3x}$
3. **Conditions** $A + B = \frac{9}{2}$, $-\frac{A}{4} + 3B + 6 = 0$: $A = 6$, $B = -\frac{3}{2}$

Decision

$y = 6e^{-x/4} - \frac{3}{2}e^{3x} + 6xe^{3x}$

Maclaurin: match at a point

$$f(x) = \sum \frac{f^{(n)}(0)}{n!} x^n$$

- Taylor: powers of $(x - a)$.
- Accurate near the centre.

Worked: a series solution about x = 3

1. **y"(3)** $4y' + 5y + e^{2/3} = 6 + e^{2/3}$
2. **y"'(3)** $4y'' + 5y' + \frac{2}{9}e^{2/3} = 19 + \frac{38}{9}e^{2/3}$
3. **Series** divide by $2!$ and $3!$

Decision

$2 - (x-3) + (3 + \frac{1}{2}e^{2/3})(x-3)^2 + (\frac{19}{6} + \frac{19}{27}e^{2/3})(x-3)^3$

Polar staples

Cardioid $r = 2(1 + \cos\theta)$: area 6π

- $x = r\cos\theta$, $y = r\sin\theta$
- Area $= \frac{1}{2}\int r^2\, d\theta$

Worked: the lemniscate

1. **One loop** $\frac{1}{2}\int_{-\pi/4}^{\pi/4} 8\cos 2\theta \, d\theta$
2. **Integrate** $4\left[\frac{1}{2}\sin 2\theta\right]_{-\pi/4}^{\pi/4} = 4$
3. **Both loops** 2×4

Decision

Total area 8

Check yourself

Question	**Answer**
Banned in an inequality?	multiplying by an unknown sign
$\lvert z-a\rvert = \lvert z-b\rvert$?	**perpendicular bisector**
PI for ke^{px}, p a root?	$\lambda x e^{px}$
Polar area?	$\frac{1}{2}\int r^2\, d\theta$
Fifth roots of w: spacing?	$\frac{2\pi}{5}$