Further Pure Mathematics 1

Edexcel IAL FP1 (WFM01/01); compulsory for IAS/IAL Further Mathematics (and Pure Mathematics)

Unit FP1 at a glance

Family	Must know
quadratic roots	$\alpha+\beta=-\frac{b}{a},\ \alpha\beta=\frac{c}{a}$
series	$\sum r=\frac{1}{2}n(n+1)$
complex	$\lvert z\rvert=\sqrt{a^2+b^2},\ \overline{z}=a-bi$
conics	$y^2=4ax;\ xy=c^2$
matrices	$\det=ad-bc;\ (AB)^{-1}=B^{-1}A^{-1}$

Addition is vector addition

$$(3 + 2i) + (-1 + 2i) = 2 + 4i$$

- Plot $a + bi$ at (a, b).
- Add real parts, add imaginary parts.

Modulus and conjugate

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{|z_2|^2}$$

- $z\bar{z} = a^2 + b^2 = |z|^2$
- Divide by multiplying by the **conjugate**.

Worked: a real sum pins the imaginary part

1. **Moduli** $|z_1 z_2| = 5r = 15$, so $r = 3$ and $z_1 = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$
2. **Real sum** $z_1 + z_2$ real forces $z_2 = a + \frac{3}{2}i$
3. **Modulus 5** $a^2 + \frac{9}{4} = 25$, so $a = \pm\frac{\sqrt{91}}{2}$

Decision

$z_2 = \pm\frac{\sqrt{91}}{2} + \frac{3}{2}i$: two answers, mirror images.

Conjugate roots come in pairs

$$(z - z_1)(z - \bar{z}_1) = z^2 - 2\,\mathrm{Re}(z_1)z + |z_1|^2$$

- Real coefficients only.
- A real cubic: three real, or one plus a pair.

Worked: a quartic with a complex root

1. **Factor** sum 6, product 34: $z^2 - 6z + 34$
2. **Compare** $34c = 68$, $c = 2$; $34b - 12 = 192$, $b = 6$
3. **Finish** $P = 7$, $Q = 204$; $7z^2 + 6z + 2 = 0$

Decision

Other roots: $z = \frac{-3 \pm i\sqrt{5}}{7}$

Sum and product, never the roots

Expression	In sum and product
$\alpha^2 + \beta^2$	$(\alpha + \beta)^2 - 2\alpha\beta$
$\alpha^3 + \beta^3$	$(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
$\frac{1}{\alpha} + \frac{1}{\beta}$	$\frac{\alpha+\beta}{\alpha\beta}$
new equation	$x^2 - (\text{sum})x + \text{product} = 0$

Worked: new roots without solving

1. **Sum** $(\alpha+\beta)^2-2\alpha\beta+(\alpha+\beta)=225+\frac{10}{3}-15=\frac{640}{3}$
2. **Product** $(\alpha^3+\beta^3)+\alpha\beta+(\alpha\beta)^2=-\frac{31040}{9}$
3. **Equation** $x^2-\frac{640}{3}x-\frac{31040}{9}=0$, times 9

Decision

$$9x^2-1920x-31040=0$$

Locate, then home in

Method	Next estimate
interval bisection	the midpoint of the sign change
linear interpolation	$a - \dfrac{(b-a)f(a)}{f(b)-f(a)}$
Newton–Raphson	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Worked: bisection and interpolation

1. **Bisection** $f(0) = 4 > 0$, $f(0.5) < 0$, $f(0.25) = -8.392 < 0$
2. **Interval** the root lies in $[0,\, 0.25]$
3. **Interpolation** $g(4) = 2.396$, $g(5) = -0.502$: $4 + \frac{2.396}{2.898} = 4.827$

Decision

Each answer backed by the evaluations that justify it.

Newton–Raphson follows the tangent

$$x_1 = 2 - \frac{-1}{10} = 2.1$$

- $f(2) = -1$, $f'(2) = 10$
- Next: $x_2 = 2.0946$

Focus and directrix

> Every point is as far from the **focus** as from the **directrix**.

- $y^2 = 4ax$: $x = at^2$, $y = 2at$
- Distances unlock the geometry.

Worked: the focus-directrix distances

1. **Equal distances** $PM = PF = 5$, $PM \perp$ **directrix**
2. **Right triangle** $QM = \sqrt{10^2 - 5^2} = 5\sqrt{3}$
3. **Both sides** M between Q and R: $QR = 2 \times 5\sqrt{3}$

Decision

$QR = 10\sqrt{3}$

Worked: a normal to the hyperbola

1. **Gradient** $xy = 81$: $\frac{dy}{dx} = -\frac{81}{x^2} = -\frac{1}{t^2}$
2. **Normal** $ty = t^3x + 9(1 - t^4)$
3. **At t = 1/3** $P(3, 27)$, $A(0, \frac{80}{3})$

Decision

Area $OPA = \frac{1}{2} \times 3 \times \frac{80}{3} = 40$

Inverse of a 2×2 matrix

$$\mathbf{M}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- Determinant zero: **singular**, no inverse.
- Inverses reverse order.

Worked: an inverse that must stay consistent

1. **Determinant** $\det \mathbf{M} = -5 - 3a$
2. **Bottom-left** $\frac{3}{5+3a} = 6$, matching $2\mathbf{M} + 8\mathbf{I}$
3. **Solve** $5 + 3a = \frac{1}{2}$

Decision

$a = -\frac{3}{2}$, and every other entry agrees.

Columns are the images of i and j

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}: \mathbf{i} \to (2,0), \mathbf{j} \to (1,1)$$

- Read the transformation off the matrix.
- $|\det|$ is the **area scale factor**.

The standard matrices

Transformation	**Matrix**
reflection in the x-axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
reflection in $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
rotation θ anticlockwise	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
enlargement, factor k	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

Worked: describing and combining

1. **Describe P** $\mathbf{i} \to (0,-1)$, $\mathbf{j} \to (1,0)$: 90° clockwise
2. **Image A** reflect $(4,3)$ in $y=-x$: $A=(-3,-4)$
3. **Undo** $B=\mathbf{P}^{-1}\mathbf{Q}^{-1}A=\frac{1}{5}(4,-3)$

Decision

$B=(0.8,\ -0.6)$

Worked: a sum to 2n

1. **Split** $2\sum r^2 - 2n$, with r up to $2n$
2. **Substitute** $\frac{2n(2n+1)(4n+1)}{3} - 2n$
3. **Factor** $\frac{4n}{3}(4n^2+3n-1) = \frac{4n}{3}(n+1)(4n-1)$

Decision

$\frac{4}{3}n(n+1)(4n-1)$; check $n=1$: $1+7=8$

Induction in three moves

> The closing sentence is part of the proof.

- Four families: sums, divisibility,
- recurrences and matrix powers.

Worked: divisibility by 3

1. **Base** $n = 1$: $1 + 2 = 3$
2. **Assume** $k^3 + 2k = 3m$
3. **Step** $(k+1)^3 + 2(k+1) = 3m + 3k^2 + 3k + 3$

Decision

$= 3(m + k^2 + k + 1)$: true for all positive integers n.

Check yourself

Question	**Answer**
Root $3+5i$, real coefficients: another root?	$3-5i$
$\alpha^2+\beta^2$ in sum and product?	$(\alpha+\beta)^2-2\alpha\beta$
PF $=5$ on a **parabola**: PM?	5
Area scale factor of $\mathbf{M}$?	$\lvert\det\mathbf{M}\rvert$
$(\mathbf{AB})^{-1}$?	$\mathbf{B}^{-1}\mathbf{A}^{-1}$