

# AQA · GCSE · Mathematics

Reviewed topic references and applicable supplements

exercises.lol

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## 1 · Number

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### Number: the foundations everything else stands on

- Order and calculate with positives, negatives, decimals and fractions; the four operations with formal written methods.
- Primes, factors, HCF/LCM by prime decomposition; indices, roots, surds and standard form (H).
- Fractions  $\leftrightarrow$  decimals  $\leftrightarrow$  percentages as operators; rounding, estimation and bounds (H).

### Arithmetic, factors, multiples and primes (N1–N5)

**Ordering and symbols:** order positive and negative integers, decimals and fractions on a number line; use  $=$ ,  $\neq$ ,  $<$ ,  $>$ ,  $\leq$ ,  $\geq$ .

**The four operations (N2):** formal written methods for integers, decimals, **proper and improper fractions and mixed numbers**, all signs – set in context including household finance: **profit, loss, cost price, selling price, debit, credit, balance, income tax, VAT, interest rate**.

**Inverse operations and priority (N3):** cancellation to simplify; **BIDMAS**

– brackets, indices/powers, roots, division/multiplication, addition/subtraction.

**Primes and structure (N4):** prime numbers, **factors, multiples, common factors, common multiples, HCF, LCM; prime factorisation** in index form (product notation; unique factorisation). Worked pattern:

- $200 = 2^3 \times 5^2$
- HCF = product of the **lowest** powers of common primes; LCM = product of the **highest** powers of all primes.

(N5) systematic listing and the **product rule for counting** (H).

### Indices, surds, standard form and bounds (N7–N9, N14–N16)

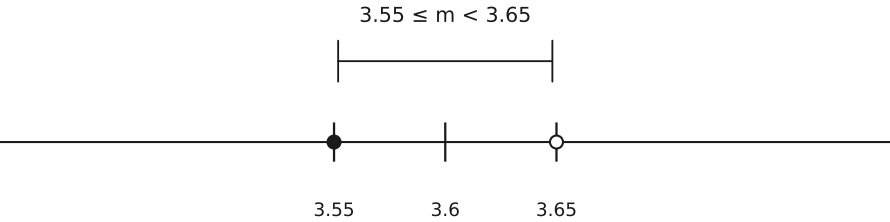
**Indices (N7):** laws –  $a^m \times a^n = a^{(m+n)}$ ,  $a^m \div a^n = a^{(m-n)}$ ,  $(a^m)^n = a^{(mn)}$ ,  $a^{(-n)} = 1/a^n$ ,  $a^{(1/2)} = \sqrt{a}$ ; (H) fractional indices  $a^{(m/n)}$  = the n-th root of  $a^m$ .

**Surds (N8, H):**  $\sqrt{12} = \sqrt{(4 \times 3)} = 2\sqrt{3}$  – simplify; **rationalise denominators** (multiply by the conjugate:  $1/(3+\sqrt{2}) \times (3-\sqrt{2})/(3-\sqrt{2}) = (3-\sqrt{2})/7$ ).

**Standard form (N9):**  $A \times 10^n$ ,  $1 \leq A < 10$ , n an integer – add/subtract (match powers), multiply (multiply coefficients and add exponents), or divide (divide coefficients and subtract exponents); interpret calculator displays.

**Estimation (N14):** round each value to 1 significant figure, compute, decide over- or

under-estimate; check calculations by approximation.



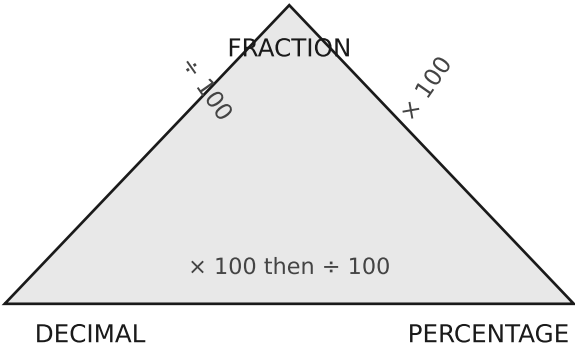
$m = 3.6$  to 1 decimal place: closed at the lower bound, open at the upper

**Rounding and error intervals** (N15): to decimal places and significant figures (know not to round mid-calculation); (H) inequality notation for error intervals – 3.6 kg to 1 d.p. means  $3.55 \leq m < 3.65$ ; truncation gives  $3.6 \leq m < 3.7$ .

**Bounds** (N16, H): upper bound of a length 9 m to the nearest metre is 9.5; in calculations, for positive inputs, an upper product bound uses upper input bounds and an upper quotient bound uses upper numerator divided by lower denominator. Check signs first; these shortcuts need not hold for negative inputs. Track strict endpoints; give the answer to an appropriate degree of accuracy (often the bound width decides).

**Fractions, decimals and percentages (N10–N13, R9 links)**

$\frac{1}{2} = 0.5 = 50\%$  — every vertex names the same number



**Conversions** (N10):  $3.5 = \frac{7}{2}$ ,  $0.375 = \frac{3}{8}$ ; (H) recurring decimal s ↔

fractions:  $0.6 = \frac{6}{9} = \frac{2}{3}$ ;  $0.45$  (pair)  $\rightarrow \frac{45}{99} = \frac{5}{11}$ . Method:  $x = 0.6$ ,  $10x = 6.6$ ,  $9x = 6$ .

**Fractions as operators** (N12): “ $\frac{3}{5}$  of 40”  $= 40 \times \frac{3}{5} = 24$ ; percentage as multiplier – increase by 15%  $= \times 1.15$ ; decrease by 20%  $= \times 0.8$ ; reverse percentage: find the original before the change by dividing by the multiplier.

**Units** (N13): metric conversions for length, area, volume, capacity; compound measures (speed, density, pressure) link to topic 3.

**Checklist before you call this topic done**

- Four operations on fractions/mixed numbers and negatives without a calculator; BIDMAS traps.
- Prime decomposition  $\rightarrow$  HCF/LCM; product rule for counting (H).
- Index laws including negative and fractional; simplify and rationalise surds (H).
- Standard form arithmetic; rounding and error intervals; upper/lower bounds through a formula (H).
- FDP conversions including recurring decimals (H); percentage multipliers and reverse percentages.

# Number – native teaching supplement

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## Supported teaching and tier boundary

8300: Number. Version: Version 1.0, 12 September 2014; first examination 2017.

Foundation teaching and Higher additions are labelled below. This reference packages the existing native-lesson crosswalk. It does not certify unreviewed specification rows or a whole qualification. Original diagnostics are separate and are not reproduced.

### ■ Exact arithmetic and estimation · Foundation

Prime factors reveal shared structure. Use the smallest common prime powers for the HCF and the largest for the LCM. Estimate before calculating; use brackets to preserve the order of operations.

$$72 = 2^3 \times 3^2, \quad 90 = 2 \times 3^2 \times 5, \quad \text{HCF} = 18$$

$72=2^3 \times 3^2$  and  $90=2 \times 3^2 \times 5$ . Their HCF is  $2 \times 9=18$ . Make 18 bags with 4 pencils and 5 pens each. Their LCM is  $2^3 \times 3^2 \times 5=360$ . A prime integer is greater than 1 and has exactly two positive factors. The number 1 is not prime; a composite positive integer greater than 1 has more than two positive factors. Every integer greater than 1 has a unique product of prime factors apart from their order. For example,  $60=2^2 \times 3 \times 5$ . A factor divides a number; a multiple is the result of multiplying it by an integer. HCF uses common smallest powers, while LCM uses largest powers.

The HCF divides both numbers; the LCM is a multiple of both. They answer different questions. A decimal estimate is not an exact fraction.

For a non-calculator paper, keep fractions exact and show cancellation. For a calculator paper, enter the full expression and compare with your estimate.

### ■ Exact arithmetic and estimation · Higher

Prime factors reveal shared structure. Use the smallest common prime powers for the HCF and the largest for the LCM. Estimate before calculating; use brackets to preserve the order of operations.

$$a = \prod p_i^{\alpha_i}, \quad b = \prod p_i^{\beta_i}, \quad \text{HCF}(a, b) = \prod p_i^{\min(\alpha_i, \beta_i)}$$

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$$72 = 2^3 \times 3^2$$

$$90 = 2 \times 3^2 \times 5$$

Shared smallest powers:  $2 \times 3^2 = 18$

*Original native-lesson illustration; labels belong to its worked example.*

### ■ Integer indices and standard form · Foundation

Use integer powers, square and cube roots and standard form with  $1 \leq a < 10$ . In multiplying powers with the same base, add indices; in division, subtract them.

$$a^m a^n = a^{m+n}$$

$0.000072=7.2 \times 10^{-5}$ . Also  $2^3 \times 2^4=2^7=128$ . The square root of 81 is 9. Check a standard-form answer by writing it out as a decimal. Useful powers include  $3^3=27$ ,  $4^3=64$ ,  $5^3=125$ ,  $15^2=225$  and  $10^6=1,000,000$ . Integer laws give  $2^0=1$ ,  $2^{-3}=1/8$  and  $(2^3)^2=2^6=64$ . For standard-form multiplication,  $(3 \times 10^5)(4 \times 10^{-3})=12 \times 10^2=1.2 \times 10^3$ . Division gives  $(6 \times 10^5)/(2 \times 10^2)=3 \times 10^3$ . For addition, first align exponents:  $3 \times 10^4 + 2 \times 10^3=3.2 \times 10^4$ .

Index laws do not turn a sum into a single power:  $2^3 + 2^4=24$ , not  $2^7$ . Do not round a surd when an exact answer is requested.

This Foundation/Core lesson excludes fractional powers and surd rationalisation. Estimate a result before using a calculator and retain the required precision.

■ Indices, surds and standard form · Higher

For the same positive base, multiplication adds indices and division subtracts them. A negative index means reciprocal; a fractional index represents a root. Standard form has  $1 \leq a < 10$ .

$$a^m a^n = a^{m+n}, \quad a^{-n} = \frac{1}{a^n}, \quad a^{m/n} = (\sqrt[n]{a})^m$$

$0.000072 = 7.2 \times 10^{-5}$ . Also  $16^{3/4} = (16^{1/4})^3 = 2^3 = 8$ . Simplify  $\sqrt{72} = 6\sqrt{2}$ , then rationalise  $1/\sqrt{2} = \sqrt{2}/2$ . Useful powers include  $3^3=27$ ,  $4^3=64$ ,  $5^3=125$ ,  $15^2=225$  and  $10^6=1,000,000$ . Integer laws give  $2^0=1$ ,  $2^{-3}=1/8$  and  $(2^3)^2=2^6=64$ . For standard-form multiplication,  $(3 \times 10^5)(4 \times 10^{-3}) = 12 \times 10^2 = 1.2 \times 10^3$ . Division gives  $(6 \times 10^5)/(2 \times 10^2) = 3 \times 10^3$ . For addition, first align exponents:  $3 \times 10^4 + 2 \times 10^3 = 3.2 \times 10^4$ . For a positive base,  $a^{(m/n)} = (\text{the } n\text{th root of } a)^m$ . Thus  $27^{(2/3)} = 3^2 = 9$  and  $16^{(-1/2)} = 1/4$ . To estimate  $\sqrt{20}$  without a calculator,  $4^2 < 20 < 5^2$  gives  $4 < \sqrt{20} < 5$ ; testing  $4.5^2 = 20.25$  shows  $\sqrt{20}$  is just below 4.5. Do not use a rough estimate as an exact surd answer.

Index laws do not turn a sum into a single power:  $2^3 + 2^4 = 24$ , not  $2^7$ . Do not round a surd when an exact answer is requested.

Check powers of ten against the original quantity. Use surds for exact geometry, and round only the final length when the question asks for a decimal.

$0.000072 = 7.2 \times 10^{-5}$

7.2 is the coefficient; -5 is the power

The coefficient is between 1 and 10

*Original native-lesson illustration; labels belong to its worked example.*

■ Accuracy, bounds and compound measures · Higher

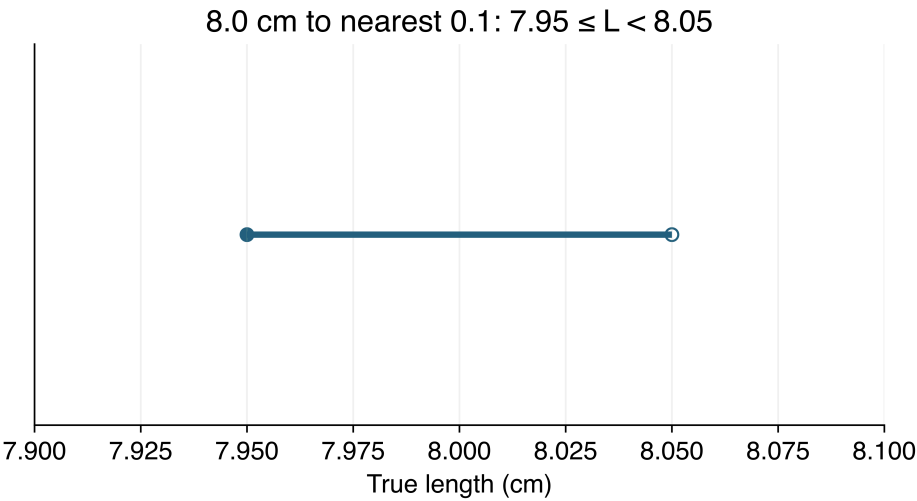
A value rounded to the nearest unit  $u$  lies from stated value- $u/2$  up to, but usually not including, stated value+ $u/2$ . For positive quantities, combine extremes according to the operation.

$$A = LW, \quad v = \frac{d}{t}$$

The lengths satisfy  $7.95 \leq L < 8.05$  and  $4.95 \leq W < 5.05$ . Since  $A = LW$ ,  $39.3525 \leq A < 40.6525$ . For speed  $d/t$ , the largest speed uses the largest distance and smallest positive time.

An upper bound is not automatically achieved. Dividing upper distance by upper time does not give the largest speed. Keep enough digits in intermediate calculations.

Distinguish measurement uncertainty from arithmetic rounding. A sensible reported precision cannot be finer than the measurements justify.



*Original native-lesson illustration; labels belong to its worked example.*

■ Signed numbers, place value and operation order · Foundation

Larger numbers lie farther right on the number line. Adding a negative moves left; subtracting a negative moves right. Multiply or divide signs first, then magnitudes. Evaluate brackets, powers and roots before multiplication/division, then addition/subtraction, working left to right within equal priority.

$$-12 + 35 - 9 = 14, \quad 18 \div 3 \times 2 = 12$$

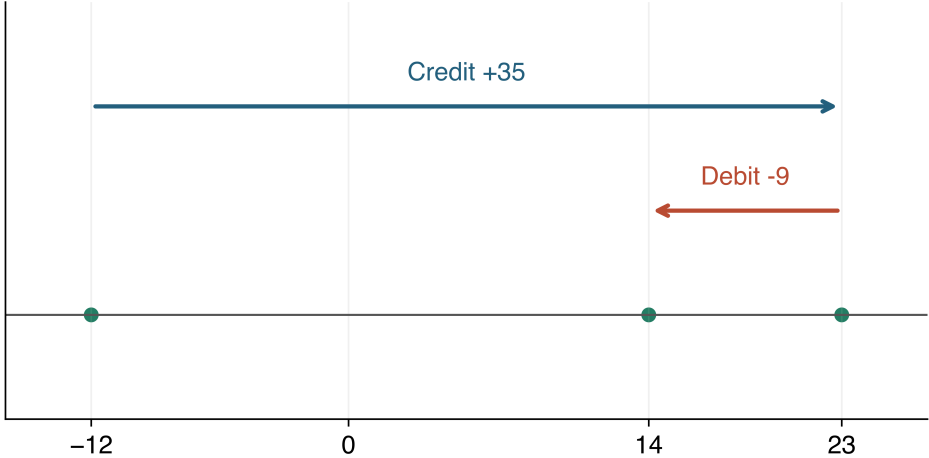
The balance is  $-12 + 35 - 9 = 14$  yuan. Also  $-4 - (-7) = 3$  and  $(-6) \times (-3) = 18$ . The reciprocal

of -4 is  $-\frac{1}{4}$ , since their product is 1. For  $18 \div 3 \times 2$ , work left to right to obtain 12. In 3.047, the 4 means four hundredths and the 7 means seven thousandths. For decimal multiplication, first calculate  $24 \times 35$ :  $24 \times 30 + 24 \times 5 = 720 + 120 = 840$ . The original  $2.4 \times 0.35$  has three decimal places in total, giving 0.840. For  $5.04 \div 0.12$  multiply both numbers by 100 to get  $504 \div 12$ ;  $12 \times 40 = 480$  leaves 24, so the answer is 42. To order negative fractions,  $-\frac{3}{4} = -0.75$  and  $-\frac{4}{5} = -0.8$ , hence  $-\frac{4}{5} < -\frac{3}{4}$ . The symbols  $\leq$  and  $\geq$  include equality;  $\neq$  means unequal.

Subtraction is not commutative: 3-8 and 8-3 differ. A negative sign outside a square is not inside its base:  $-3^2 = -9$ , but  $(-3)^2 = 9$ . Zero has no reciprocal.

Use a signed starting balance, a positive credit and a negative debit. A shop buying for 48 yuan and selling for 60 makes 12 yuan profit; reversing the prices gives a 12-yuan loss. Estimate the sign before calculating.

The balance moves from -12 to +14



Original native-lesson illustration; labels belong to its worked example.

■ Exact fractions and mixed-number operations · Foundation

Convert mixed numbers to improper fractions. Add or subtract using a common denominator; multiply numerators and denominators; divide by a nonzero fraction by multiplying its reciprocal. Cancel common factors, not added terms. These rules also apply to negative fractions.

$$\frac{3}{4} \div \frac{5}{8} = \frac{3}{4} \times \frac{8}{5} = \frac{6}{5}$$

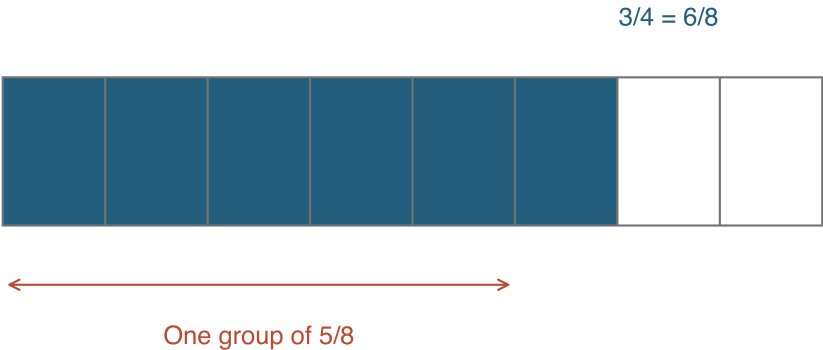
$1\frac{1}{2} - \frac{3}{4} = \frac{6}{4} - \frac{3}{4} = \frac{3}{4}$ . Also  $(-\frac{2}{3}) \times (\frac{9}{4}) = -\frac{18}{12} = -\frac{3}{2}$  and  $(\frac{3}{4}) \div (\frac{5}{8}) = (\frac{3}{4}) \times (\frac{8}{5}) = \frac{6}{5}$ . A common denominator gives  $\frac{5}{6} + \frac{3}{4} = \frac{10}{12} + \frac{9}{12} = \frac{19}{12}$ . Check division by mul-

tiplying  $\frac{6}{5}$  by  $\frac{5}{8}$  to recover  $\frac{3}{4}$ . Exact multiples of  $\pi$  follow ordinary arithmetic:  $3\pi + 2\pi = 5\pi$  and  $6\pi/3 = 2\pi$ . Leave an answer such as  $5\pi$  exact when requested;  $\pi \approx 3.14$  would introduce approximation. For subtraction,  $\frac{5}{6} - \frac{3}{4} = \frac{10}{12} - \frac{9}{12} = \frac{1}{12}$ ; for a negative mixed number,  $-1\frac{1}{2}$  means  $-(1 + \frac{1}{2}) = -\frac{3}{2}$ .

Adding denominators does not preserve the unit size:  $\frac{1}{2} + \frac{1}{3}$  is not  $\frac{2}{5}$ . Cancel only factors of a whole numerator and denominator. A division by zero is undefined.

For a non-calculator question show the common denominator or reciprocal step. Convert  $\frac{19}{12}$  to  $1\frac{7}{12}$  if a mixed number is requested; round only when the question explicitly needs a decimal.

Six eighths contain  $\frac{6}{5}$  groups of five eighths



Original native-lesson illustration; labels belong to its worked example.

■ Systematic lists and possibility grids · Foundation

Fix one first choice and list every allowed second choice before moving to the next first choice. Use a table to check that no outcome is missing or duplicated. If combinations are forbidden, remove those entries from the list.

$$\{TA, TB, TM, JA, JB, JM\}$$

List tea-apple, tea-banana, tea-melon, juice-apple, juice-banana, juice-melon: six choices. Removing tea-melon leaves five. The codes using 1,2,3 once each are 123,132,213,231,312,321: six. These answers follow from complete lists.

State whether order matters and whether repetition is allowed. Choosing A then B can be different from B then A for a code, but not for an unordered two-person team. A restriction can make a simple product invalid.

Foundation should justify answers with a complete ordered list or grid. Higher may compress a verified structure using the product rule, but must still check restrictions.

■ **Systematic lists and the product rule** · Higher

Fix one first choice and list every permitted second choice before moving to the next first choice. A table gives the same structure. When every one of m first choices allows n second choices, the Higher product rule gives mn outcomes. If restrictions change the options, count the allowed rows separately.

$$\{TA, TB, TM, JA, JB, JM\}$$

The list is tea-apple, tea-banana, tea-melon, juice-apple, juice-banana, juice-melon: six outcomes. If tea-melon is unavailable, five remain. For a three-digit code with digits 1,2,3 and no repeats, choose 3 then 2 then 1 possibilities, giving six codes: 123,132,213,231,312,321.

State whether order matters and whether repetition is allowed. Choosing A then B can be different from B then A for a code, but not for an unordered two-person team. A restriction can make a simple product invalid.

Foundation should justify answers with a complete ordered list or grid. Higher may compress a verified structure using the product rule, but must still check restrictions.

A complete grid lists six choices once each

Juice + Apple

Juice + Banana

Juice + Melon

Tea + Apple

Tea + Banana

Tea + Melon

Original native-lesson illustration; labels belong to its worked example.

■ **Exact surds and rationalising denominators** · Higher

Extract square factors:  $\sqrt{(a^2b)}=a\sqrt{b}$  for  $a\geq 0, b\geq 0$ . Add only matching root parts. Multiply roots with nonnegative radicands. To rationalise a denominator, multiply numerator and denominator by the same suitable root or conjugate.

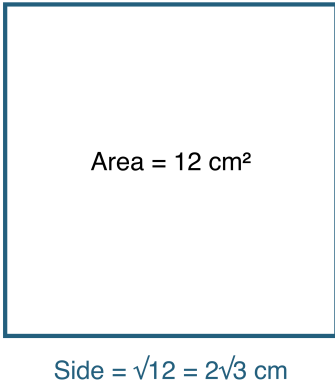
$$\sqrt{12} + \sqrt{27} = 5\sqrt{3}, \quad \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

The side is  $\sqrt{12}=\sqrt{(4\times 3)}=2\sqrt{3}$  cm. Thus  $\sqrt{12}+\sqrt{27}=2\sqrt{3}+3\sqrt{3}=5\sqrt{3}$ . Also  $6/\sqrt{3}=6\sqrt{3}/3=2\sqrt{3}$ . For  $1/(2+\sqrt{3})$ , multiply by  $(2-\sqrt{3})/(2-\sqrt{3})$ : the denominator becomes  $4-3=1$ , giving  $2-\sqrt{3}$ . A circle of radius 3 has exact area  $9\pi$ ; a decimal is an approximation.

$\sqrt{(a+b)}$  generally differs from  $\sqrt{a}+\sqrt{b}$ . Match the radicand before collecting terms. Rationalising changes the form, not the value; multiplying only the denominator changes the value.

AQA N8 Higher requires exact surds and rationalisation. Foundation retains exact fractions and multiples of  $\pi$ ; do not assign this surd lesson to Foundation.

An exact root describes the square



Original native-lesson illustration; labels belong to its worked example.

■ **Terminating decimals and fractions** · Foundation

Use the place value of the last digit to write a power-of-ten denominator, then simplify. Compare numbers using matching decimal places or a common denominator.

$$0.375 = \frac{375}{1000} = \frac{3}{8}$$

$0.375=375/1000=3/8$ . Also  $3.5=35/10=7/2$  and  $0.06=6/100=3/50$ . To order  $3/8$  and  $0.4$ , write  $0.375$  and  $0.400$ ; therefore  $3/8<0.4$ .

Keep the place value:  $0.06$  is six hundredths, not six tenths. A fraction may exceed 1. Simplify the entire numerator and denominator by the same common factor.

AQA N10 Foundation covers terminating decimals and fractions, including ordering. Recurring-decimal conversion is reserved for the Higher lesson.

■ **Terminating and recurring decimal conversions · Higher**

A finite decimal uses a power-of-ten denominator before simplifying. For a recurring decimal, multiply by powers of ten so the repeated tails align, then subtract. Use matching decimal places to compare numbers. A displayed rounded decimal need not be the exact fraction.

$$100x - x = 27, \quad x = \frac{27}{99} = \frac{3}{11}$$

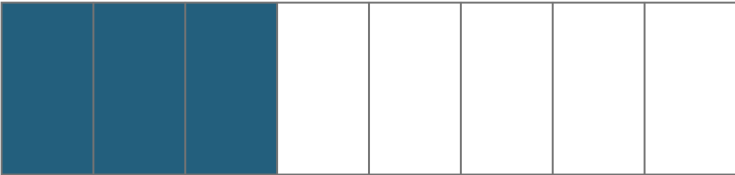
$0.375=375/1000=3/8$ . For  $x=0.272727\dots$ ,  $100x=27.272727\dots$ , so  $99x=27$  and  $x=27/99=3/11$ . For  $y=0.16666\dots$ ,  $100y-10y=16.666\dots-1.666\dots=15$ , so  $y=15/90=1/6$ . In a reduced fraction, a denominator containing only factors 2 and 5 gives a terminating decimal.

Align the recurring tails before subtracting.  $0.333333$  is finite and differs from  $0.333333\dots$ . A non-recurring prefix needs a second power of ten; blindly dividing every digit block by 99 fails.

AQA N10 Higher includes recurring conversion. Both tiers convert and order terminating decimals and fractions. Check a conversion by long division; 3 divided by 8 gives  $0.375$ .

Different exact forms, the same shaded fraction

$$3/8 = 0.375 = 375/1000$$



*Original native-lesson illustration; labels belong to its worked example.*

■ **Metric and compound-unit conversions · Foundation**

Convert each dimension. Since  $1\text{ m}=100\text{ cm}$ ,  $1\text{ m}^2=10,000\text{ cm}^2$  and  $1\text{ m}^3=1,000,000\text{ cm}^3$ . Also  $1\text{ litre}=1000\text{ cm}^3$ ,  $1\text{ kg}=1000\text{ g}$  and  $1\text{ hour}=3600\text{ seconds}$ . For a compound unit convert numerator and denominator, keeping the physical quantity unchanged.

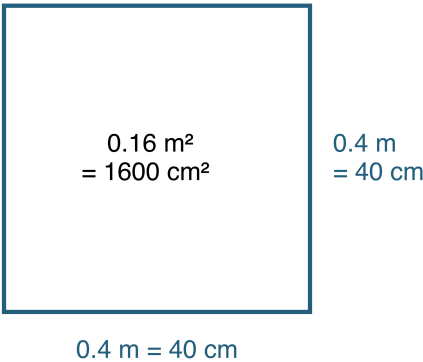
$$1\text{ m}^2 = 100^2\text{ cm}^2, \quad 72\frac{\text{km}}{\text{h}} = 20\frac{\text{m}}{\text{s}}$$

$0.4\text{ m}=40\text{ cm}$ , so the tile area is  $40\times40=1600\text{ cm}^2$ , agreeing with  $0.16\times10,000$ . A  $0.002\text{ m}^3$  container holds  $2000\text{ cm}^3=2\text{ litres}$ . A speed of  $72\text{ km/h}$  is  $72,000/3600=20\text{ m/s}$ . Write the units at each stage so the conversion can be checked.

The area multiplier is the square of the length multiplier; the volume multiplier is its cube. An hour is 60 minutes, not 100. Converting only the numerator of a speed gives an inconsistent unit.

AQA N13 and R1 include metric length, area, volume, capacity and compound measures. Use given conversion factors for unfamiliar imperial units; do not guess them.

Convert both dimensions before multiplying



*Original native-lesson illustration; labels belong to its worked example.*

■ **Estimation, rounding and simple error intervals · Foundation**

For a rough check use easy nearby numbers:  $50\times20\div10=100$ . Decimal places count digits after the point; significant figures start at the first nonzero digit. Round only the final result. Nearest-unit rounding has an interval extending half a unit either way; positive truncation keeps values from the stated value up to the next unit.

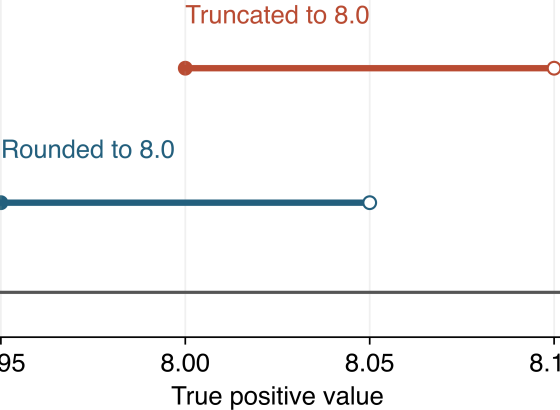
$$7.95 \leq L < 8.05$$

The calculation is about 96.18, consistent with the estimate 100. The number 0.004786 rounds to 0.0048 at 2 significant figures, but to 0.005 at 3 decimal places. If length L rounds to 8.0 cm at 1 decimal place,  $7.95 \leq L < 8.05$ . If a positive value is truncated to 8.0 at 1 decimal place,  $8.0 \leq L < 8.1$  instead. Reported precision must fit the question. Seventeen items packed six per box need three whole boxes, since two boxes hold only twelve items. Rounding  $17/6$  to two boxes would fail the physical requirement. For a nearest-0.1 reading of 8.0, each possible value differs from the report by at most 0.05; a claim of 8.08 lies outside the interval. Carry full calculator precision until a final money, length or accuracy requirement is applied.

Zeros before the first nonzero digit do not count as significant figures. Rounding and truncation give different intervals. Do not turn an approximate check into an exact answer, or round every intermediate result.

AQA N14–N16 require accuracy interpretation at both tiers. Foundation uses simple intervals and limits of accuracy. Higher combines upper and lower bounds for calculated quantities in the separate bounds lesson.

Closed left endpoint; open right endpoint



*Original native-lesson illustration; labels belong to its worked example.*

■ Ratio shares and fraction operators · Foundation

Add the ratio parts to find the whole: 2:5 has 7 equal parts. Divide the total by 7, then multiply by the required part count. A fraction acts as a multiplier; a percentage p acts as p/100. Keep part-to-part and part-to-whole comparisons separate.

$$84 \times \frac{2}{2+5} = 24$$

Each part is  $84/7=12$  yuan, giving shares 24 and 60. The first team has  $2/7$  of the whole, while its share is  $2/5$  of the other share. A  $3/4$  portion of 28 is 21. A 120% amount of 35 is  $1.2 \times 35=42$ , so percentages may exceed 100.

A ratio 2:5 does not give the first share as  $2/5$  of the total. The total has 7 parts. A multiplier over 1 increases a positive quantity; it does not automatically mean a probability.

AQA N11/N12 and R3–R8 link ratio parts to fractions and operators. Check the shares add to the stated total and simplify a ratio by dividing every part by the same positive factor.

Ratio 2:5 has seven total parts



*Original native-lesson illustration; labels belong to its worked example.*

Original independent transfer

■ Foundation

A supplier has 84 pencils and 126 pens. Find the greatest number of identical packs using everything. State each pack’s contents. A delivery charge of 18.60 CNY is shared equally among six buyers; find each share without rounding intermediate values.

■ Foundation worked solution

$84 = 2^2 \cdot 3 \cdot 7$  and  $126 = 2 \cdot 3^2 \cdot 7$ , so the HCF is 42. Each of 42 packs has  $84/42 = 2$  pencils and  $126/42 = 3$  pens. This is maximal because any pack count must divide both totals. Each delivery share is  $s = C/n = (18.60 \text{ CNY})/6 = 3.10 \text{ CNY}$ ; six shares return the original charge.

## ■ Higher

A positive rectangle has recorded length 8.0 cm and width 3.0 cm, each to the nearest 0.1 cm. Give tight lower and upper bounds for area and for length divided by width. State endpoint inclusion.

## ■ Higher worked solution

$7.95 \leq l < 8.05$  and  $2.95 \leq w < 3.05$ , in cm. Positivity makes product and quotient monotonic in the needed directions. Thus  $23.4525 \leq A < 24.5525$  in square centimetres. For the dimensionless quotient,  $7.95/3.05 < l/w < 8.05/2.95$ , or  $159/61 < l/w < 161/59$ . Neither quotient endpoint is attained, since each requires one excluded upper measurement bound. The area lower bound is attained, since both lower bounds are allowed. Blindly using “upper over lower” without sign and endpoint checks is insufficient.

## Terms

prime factor .  
index .  
lower bound .  
reciprocal .  
improper fraction .  
systematic list .  
surd .  
terminating decimal .  
recurring decimal .  
conversion factor .  
significant figure .  
ratio .

# Algebra – native teaching supplement

AQA · GCSE · Mathematics

## Supported teaching and tier boundary

8300: Algebra. Version: Version 1.0, 12 September 2014; first examination 2017.

Foundation teaching and Higher additions are labelled below. This reference packages the existing native-lesson crosswalk. It does not certify unreviewed specification rows or a whole qualification. Original diagnostics are separate and are not reproduced.

## ■ Equations, identities and rearrangement · Foundation

An equation asks which inputs satisfy an equality; an identity holds for all allowed inputs. Preserve equality by applying the same operation to both sides. State restrictions before dividing by a variable.

$$C_1 = 20 + 3x, \quad C_2 = 44 + x, \quad C_1 = C_2$$

$20+3x=44+x$  gives  $2x=24$  and  $x=12$ . Both plans then cost 56. In  $A=\pi r^2$ , divide by  $\pi$  and take the positive square root to obtain  $r=\sqrt{(A/\pi)}$ , because  $r$  is a length.

Cancelling a term is not the same as cancelling a factor. In  $(x^2+2x)/x$ , factor the numerator and retain  $x \neq 0$ . Check a rearrangement by substitution.

Define the unknown and set up a linear equation. Check the answer by substitution. Restrict this Foundation/Core lesson to simple expressions and equations.

## ■ Equations, identities and rearrangement · Higher

An equation asks which inputs satisfy an equality; an identity holds for all allowed inputs. Preserve equality by applying the same operation to both sides. State restrictions before dividing by a variable.

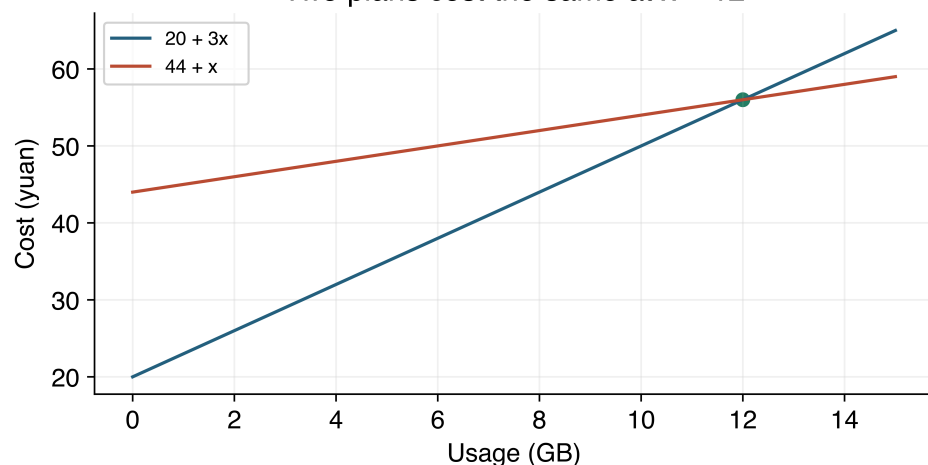
$$C_1 = 20 + 3x, \quad C_2 = 44 + x, \quad C_1 = C_2$$

$20+3x=44+x$  gives  $2x=24$  and  $x=12$ . Both plans then cost 56. In  $A=\pi r^2$ , divide by  $\pi$  and take the positive square root to obtain  $r=\sqrt{(A/\pi)}$ , because  $r$  is a length.

Cancelling a term is not the same as cancelling a factor. In  $(x^2+2x)/x$ , factor the numerator and retain  $x \neq 0$ . Check a rearrangement by substitution.

Set up the equation from units and the meaning of the unknown. A negative or fractional solution may be algebraically correct but impossible for a count.

Two plans cost the same at  $x = 12$



Original native-lesson illustration; labels belong to its worked example.

### ■ Factorising and solving simple quadratics · Foundation

Expand brackets and factorise simple quadratics. Solve by setting each factor equal to zero, and use a graph to interpret the roots.

$$(x - 3)(x - 7) = 0$$

$x^2 - 10x + 21 = (x - 3)(x - 7)$ . Hence the equation  $x^2 - 10x + 21 = 0$  has roots 3 and 7. Check each root by substitution and mark both intercepts on the graph.

Multiplying an inequality by a negative number reverses its direction. A sketch must show which side of each root satisfies the inequality. Geometry may restrict  $x$  further.

This Foundation/Core lesson uses factorisation and graphical roots; the discriminant, quadratic formula and quadratic inequalities are reserved for the advanced tier.

### ■ Quadratics and inequalities · Higher

Factor where possible; otherwise complete the square or use the quadratic formula. A quadratic inequality needs the sign on intervals, not only the roots. The discriminant identifies repeated or missing real roots.

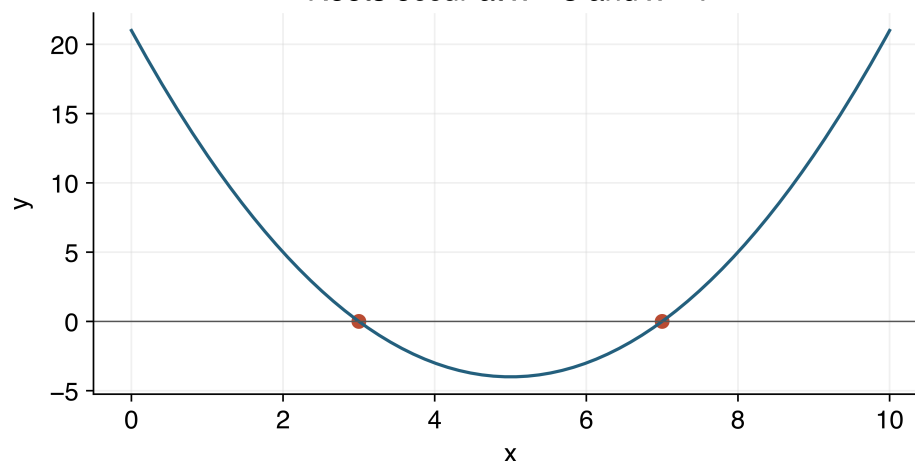
$$ax^2 + bx + c = 0, \quad \Delta = b^2 - 4ac$$

The equation  $x^2 - 10x + 21 = 0$  factorises as  $(x - 3)(x - 7) = 0$ , giving roots 3 and 7. Complete the square:  $x^2 - 10x + 21 = (x - 5)^2 - 4$ , giving turning point (5, -4) and symmetry line  $x = 5$ . The formula  $x = [-b \pm \sqrt{(b^2 - 4ac)}] / (2a)$  also gives  $(10 \pm 4) / 2 = 3, 7$ . For  $2x^2 + 5x - 3 = 0$ ,

the discriminant is 49 and roots are  $(-5 \pm 7) / 4 = 1/2, -3$ . A graph gives approximate roots when exact factorisation is inconvenient. For an enclosure with area  $A = x(10 - x)$ , complete the square to get  $A = 25 - (x - 5)^2$ . The greatest area is 25 at  $x = 5$ , within  $0 < x < 10$ .

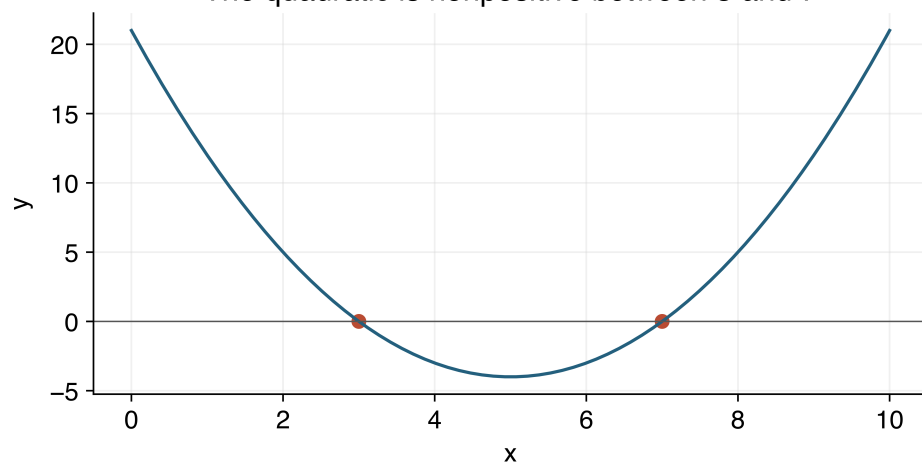
Multiplying an inequality by a negative number reverses its direction. A sketch must show which side of each root satisfies the inequality. Geometry may restrict  $x$  further. AQA A11/A18 Higher interprets roots, intercepts and turning points, completes the square and uses the quadratic formula, including equations needing rearrangement. Factorisation and substitution check each result.

Roots occur at  $x = 3$  and  $x = 7$



Original native-lesson illustration; labels belong to its worked example.

The quadratic is nonpositive between 3 and 7



Original native-lesson illustration; labels belong to its worked example.

## ■ Two linear simultaneous equations · Foundation

Multiply equations to make a variable cancel, or substitute an expression from one equation into the other. Solve the remaining linear equation and recover the second variable. The intersection is a point satisfying both original equations.

$$x + y = 12, \quad 3x + 2y = 31$$

For  $x+y=12$  and  $3x+2y=31$ , subtract twice the first equation from the second:  $x=7$ . Then  $y=5$ . Check both  $7+5=12$  and  $3 \times 7 + 2 \times 5 = 31$ . The two straight-line graphs meet at  $(7,5)$ .

Check the ordered pair in both original equations. Multiplying an equation means multiplying every term, including the constant. Parallel distinct lines have no common solution.

AQA A19 Foundation solves two linear equations by elimination or substitution and interprets the graph intersection. Linear/quadratic systems belong to Higher.

## ■ Linear and linear/quadratic simultaneous equations · Higher

For two linear equations, use elimination or substitution and check both equations. For a line and a quadratic, substitute the linear relation first; then solve the resulting quadratic. For inequalities, shade the region satisfying every condition.

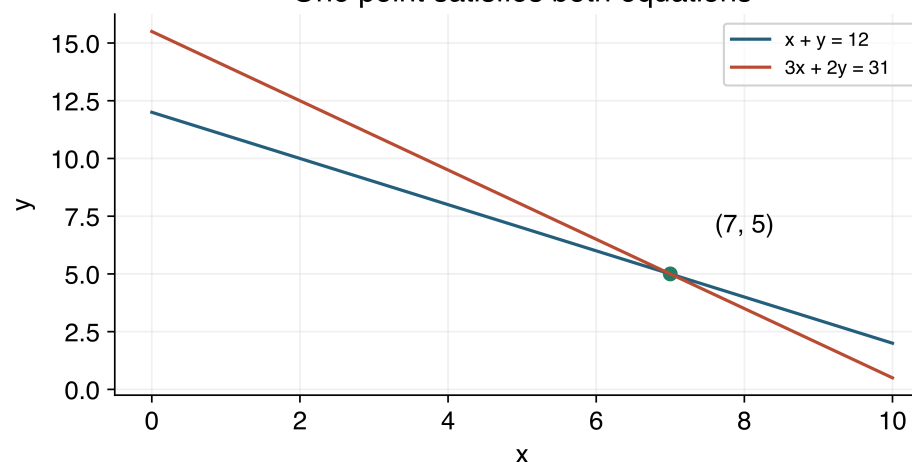
$$x + y = 12, \quad 3x + 2y = 31$$

For  $x+y=12$  and  $3x+2y=31$ , subtract twice the first equation to obtain  $x=7, y=5$ . For  $y=x+2$  and  $y=x^2$ , equate outputs:  $x^2-x-2=0$ , hence  $x=2$  or  $-1$ . The intersections are  $(2,4)$  and  $(-1,1)$ , and both satisfy the line and parabola.

One equation checked is not enough. A line can meet a quadratic twice, so retain both solutions unless the context removes one. Inequality boundaries may be included or excluded according to the sign.

AQA A19 Higher includes two linear equations and linear/quadratic systems. Elimination suits linear pairs; substitution reduces a line/curve pair to a quadratic. Retain every solution and check both original equations.

One point satisfies both equations



Original native-lesson illustration; labels belong to its worked example.

## ■ Straight lines and gradients · Foundation

Gradient is change in  $y$  divided by change in  $x$ . A straight line has  $y=mx+c$ , where  $c$  is its  $y$ -intercept. Parallel lines have equal gradients.

$$y = mx + c$$

Through  $(2,5)$  with gradient 3, substitute to get  $5=3 \times 2 + c$ , so  $c=-1$  and  $y=3x-1$ . Points  $(1,2)$  and  $(4,8)$  give gradient  $(8-2)/(4-1)=2$ .

A vertical line has no finite gradient; do not force it into  $y=mx+c$ . Read the signs of a circle's centre carefully. The radius to a tangent is perpendicular to the tangent.

Plot a straight line using two checked points and label its intercept. Perpendicular-gradient formulae and circle equations are not part of this Foundation/Core lesson.

## ■ Coordinate geometry and tangents · Higher

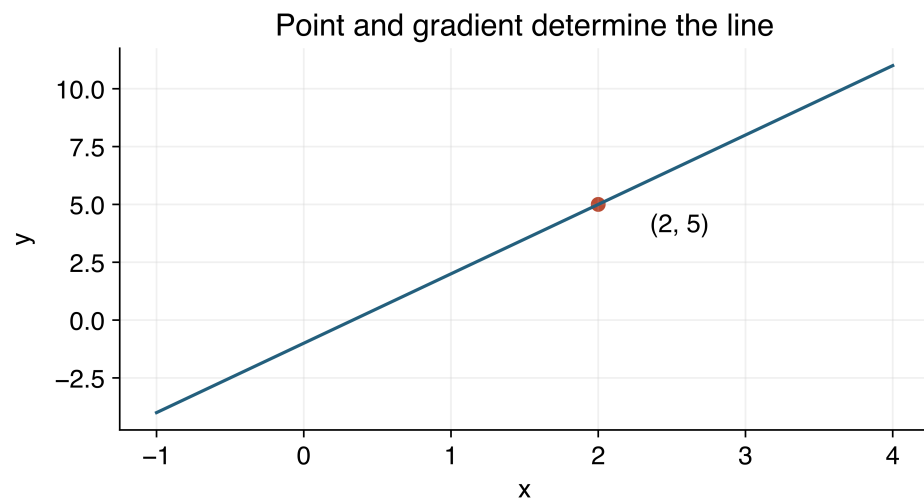
A line through  $(x_1, y_1)$  with gradient  $m$  has  $y - y_1 = m(x - x_1)$ . Parallel lines have equal gradients. Finite perpendicular gradients multiply to  $-1$ . A circle has  $(x - a)^2 + (y - b)^2 = r^2$ .

$$y - y_1 = m(x - x_1), \quad (x - a)^2 + (y - b)^2 = r^2$$

Through  $(2, 5)$  with gradient  $3$ ,  $y - 5 = 3(x - 2)$ , so  $y = 3x - 1$ . A perpendicular through the same point has  $y - 5 = -(x - 2)/3$ . The circle  $(x - 2)^2 + (y + 1)^2 = 25$  has centre  $(2, -1)$  and radius  $5$ .

A vertical line has no finite gradient; do not force it into  $y = mx + c$ . Read the signs of a circle's centre carefully. The radius to a tangent is perpendicular to the tangent.

Before solving a line-circle intersection, predict whether there are zero, one or two intersections. Substitution produces a quadratic whose discriminant checks the prediction.



*Original native-lesson illustration; labels belong to its worked example.*

## ■ Arithmetic sequences and nth terms · Foundation

Find a constant difference for an arithmetic sequence. Its  $n$ th term is  $a + (n - 1)d$ . A term-to-term rule describes how to reach the next term; a position-to-term rule gives a term directly.

$$u_n = a + (n - 1)d$$

For  $5, 8, 11, 14, \dots$  the common difference is  $3$ . The  $n$ th term is  $5 + 3(n - 1) = 3n + 2$ . At  $n = 8$ ,  $u_8 = 26$ . To find the position of  $62$ , solve  $3n + 2 = 62$ , giving  $n = 20$ .

The first term has index  $1$ , so the exponent is  $n - 1$ . A sequence is a list; a series is a sum. A geometric sequence can alternate in sign and still converge.

Generate several terms and check a proposed  $n$ th-term rule. Infinite geometric series and advanced sum formulae are excluded from this Foundation/Core lesson.

## ■ Arithmetic and finite geometric sequences · Higher

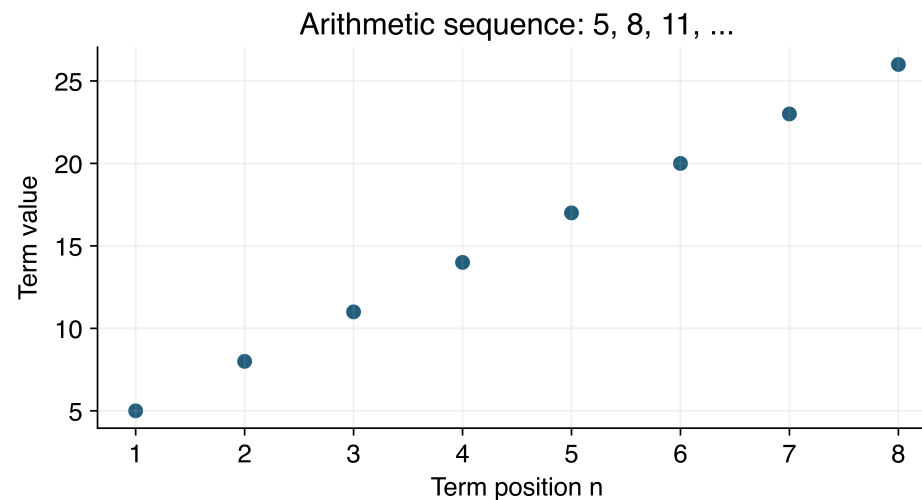
An arithmetic sequence adds a constant difference and has  $n$ th term  $a + (n - 1)d$ . A geometric sequence multiplies by a constant ratio and has  $n$ th term  $ar^{n-1}$ . Check the starting position and keep a surd ratio exact when one is supplied.

$$u_n = a + (n - 1)d, \quad v_n = ar^{n-1}$$

For  $5, 8, 11, \dots$  the  $n$ th term is  $3n + 2$ , so  $u_8 = 26$ . For  $2, 6, 18, \dots$  the common ratio is  $3$  and  $u_n = 2 \times 3^{n-1}$ , giving  $u_4 = 54$ . The geometric sequence  $1, \sqrt{2}, 2, 2\sqrt{2}, \dots$  has ratio  $\sqrt{2}$ . These are finite-term calculations; no infinite-series sum is used.

A geometric ratio is not a common difference. The exponent is  $n - 1$  when  $a$  is the first term at position  $1$ . A finite pattern does not itself justify an infinite-sum formula.

AQA A23–A25 Higher uses arithmetic rules, geometric progression patterns and quadratic  $n$ th terms. Arithmetic-series formulae and infinite-series sums are excluded from this GCSE lesson.



*Original native-lesson illustration; labels belong to its worked example.*

■ Algebraic notation, substitution and vocabulary · Foundation

A term is a part joined by addition or subtraction. In  $4+3d$ , 4 is a constant and 3 is the coefficient of d. The expression has no equality sign;  $4+3d=19$  is an equation. A formula connects named quantities. In  $ab$ , multiplication is understood;  $a^2b$  means  $a \times a \times b$ , not  $a \times b \times b$ .

$$a^2b = (-2)^2 \times 3 = 12$$

For  $d=5$ , the charge is  $4+3 \times 5=19$  yuan. For  $a=-2, b=3$ ,  $a^2b=(-2)^2 \times 3=12$ . The fraction coefficient in  $(3/4)x$  gives 6 when  $x=8$ . An inequality  $4+3d \leq 19$  describes all permitted distances, while an identity such as  $2(x+3)=2x+6$  is true for every  $x$ .

Put a negative substituted number in brackets before squaring. A coefficient is not an exponent. An expression can be evaluated but cannot be solved unless a condition or equation is supplied.

AQA A1–A3 require precise notation and vocabulary, including formulae from other subjects. Identify the inputs and units before substitution; keep fraction coefficients exact.

Constant + variable charge



*Original native-lesson illustration; labels belong to its worked example.*

■ Collecting, expanding and factorising expressions · Foundation

Collect like terms, distribute over brackets and reverse expansion by factorising. Take out a common factor first. Expand products of two binomials and factorise simple monic quadratics and differences of squares.

$$(x + 2)(x + 3) = x^2 + 5x + 6$$

$3x+4x=7x$ ;  $3x+4y$  cannot be combined. Expand  $2(x+3)=2x+6$  and  $(x+2)(x+3)=x^2+5x+6$ . Reverse the last identity to factorise  $x^2+5x+6$ . Difference of squares gives  $x^2-9=(x-3)(x+3)$ . The same expansion rule works with given roots:  $(\sqrt{2}+1)(\sqrt{2}-1)=2-1=1$ .  $x^2$  and  $x$  are unlike terms. A factor multiplies a whole expression; it is not a separate added term. Check a proposed factorisation by expanding it.

AQA A4 Foundation includes collecting, single brackets, common factors, two-binomial expansion and monic quadratic factorisation. Higher adds non-monic factorisation, more binomials, surds and algebraic fractions.

■ Collecting, expanding and factorising expressions · Higher

Collect only like terms. Distribute multiplication over every term in a bracket, including signs. Factorising reverses expansion. Use common factors first; then pairs for a quadratic. Algebraic fractions may cancel common factors, with forbidden denominator values stated.

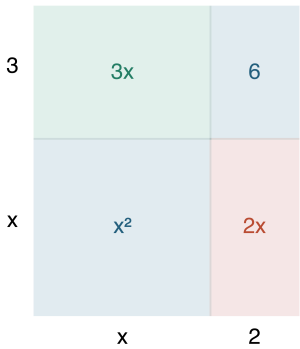
$$(x + 2)(x + 3) = x^2 + 5x + 6$$

$3x+4x=7x$ , but  $3x+4y$  cannot be combined. Expand  $2(x+3)=2x+6$  and  $(x+2)(x+3)=x^2+5x+6$ . Reverse the last result to factorise  $x^2+5x+6$ . Difference of squares gives  $x^2-9=(x-3)(x+3)$ . For Higher,  $2x^2+5x+2=(2x+1)(x+2)$ , and  $(x^2-9)/(x-3)=x+3$  only when  $x \neq 3$ . Also  $(x+1)(x+2)(x+3)=x^3+6x^2+11x+6$ .

$x^2$  and  $x$  are unlike terms. Cancelling across a sum is invalid; factor the whole expression first. A simplified fraction must retain values excluded by its original denominator.

AQA A4 Foundation includes collecting, single brackets, common factors, two-binomial expansion and monic quadratic factorisation. Higher adds non-monic factorisation, more binomials, surds and algebraic fractions.

$$(x + 2)(x + 3) = x^2 + 5x + 6$$



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Rearranging formulae and checking the subject · Foundation

Undo operations on both sides in reverse order. In  $A=bh/2$ , multiply both sides by 2 then divide by nonzero  $h$  to obtain  $b=2A/h$ . When the desired variable occurs twice, collect it before dividing. State restrictions introduced by division.

$$A = \frac{bh}{2} \iff b = \frac{2A}{h} \quad (h \neq 0)$$

For  $A=24, h=6$  in  $A=bh/2$ ,  $b=2 \times 24/6=8$ . From  $v=u+at$ ,  $a=(v-u)/t$ ;  $v=19, u=4, t=5$  gives  $a=3$ . From  $y=3x+2$ ,  $x=(y-2)/3$ . From  $p=qx+r$ ,  $x=(p-r)/q$  for  $q \neq 0$ . Substitute the result into the original equation to check it.

Doing an operation to only one side breaks the equality. The subject is a variable, not a numerical answer. Check the rearranged result by putting the found value into the original formula.

AQA A5 includes standard and given formulae from other subjects, in words and symbols. Choose a valid unit conversion before substituting; a rearrangement does not itself change units.

### ■ Rearranging formulae and checking the subject · Higher

Undo operations on both sides in reverse order. In  $A=bh/2$ , multiply both sides by 2 then divide by nonzero  $h$  to obtain  $b=2A/h$ . When the desired variable occurs twice, collect it before dividing. State restrictions introduced by division.

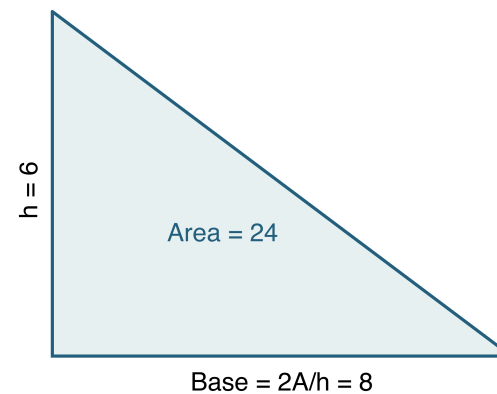
$$A = \frac{bh}{2} \iff b = \frac{2A}{h} \quad (h \neq 0)$$

For  $A=24, h=6$ ,  $b=2 \times 24/6=8$ . From  $v=u+at$ ,  $a=(v-u)/t$ ; for  $v=19, u=4, t=5$  this gives  $a=3$ . From  $y=3x+2$ ,  $x=(y-2)/3$ . From  $p=qx+r$ ,  $x=(p-r)/q$  for  $q \neq 0$ . Higher may factor repeated subjects:  $y=3x+px$  gives  $x=y/(3+p)$  when  $p \neq -3$ .

Doing an operation to only one side breaks the equality. The subject is a variable, not a numerical answer. Check the rearranged result by putting the found value into the original formula.

AQA A5 includes standard and given formulae from other subjects, in words and symbols. Choose a valid unit conversion before substituting; a rearrangement does not itself change units.

Undo the area formula to find the base



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Identities, equivalence and algebraic arguments · Foundation

An equation may hold only for some values, while an identity holds for every allowed input. Establish equivalent expressions by valid expansion or factorisation. A few matching inputs are checks rather than a general argument.

$$3(x+2) - x = 2x + 6$$

Expand  $3(x+2)-x=3x+6-x=2x+6$ . This chain of valid steps shows the expressions agree for every  $x$ . However,  $x^2=x$  holds only when  $x=0$  or  $x=1$ ; at  $x=2$ ,  $4 \neq 2$ . Evaluating  $(2n+1)^2$  at  $n=3$  gives 49, but this one calculation does not establish an all-integers claim.

Start from an expression or the assumptions, not from the conclusion as though it were already true. An example can disprove an all-values claim, but one confirming example cannot prove it.

AQA A3/A6 Foundation distinguishes expression, equation and identity and argues equivalence using algebra. General parity and divisibility proofs belong to the Higher variant.

### ■ Identities, equivalence and algebraic arguments · Higher

An equation may be true only at certain values. An identity is true at every allowed value. Establish equivalence by valid expansion or factorisation; testing a few inputs is only a check. Higher proofs use a general integer or algebraic variable and a conclusion tied to its definition.

$$3(x + 2) - x = 2x + 6$$

Expanding  $3(x+2)-x$  gives  $3x+6-x=2x+6$ , proving equivalence for every  $x$ . But  $x^2=x$  holds only for  $x=0$  or  $1$ . A counterexample  $x=2$  rejects an all-values claim. For Higher, an odd integer is  $2n+1$ ; its square is  $4n^2+4n+1=2(2n^2+2n)+1$ , so it is odd for every integer  $n$ .

Start from an expression or the assumptions, not from the conclusion as though it were already true. An example can disprove an all-values claim, but one confirming example cannot prove it.

AQA A6 Foundation distinguishes equation/identity and argues equivalence. Higher extends this to algebraic proofs. State integer restrictions when using parity or consecutive integers.

Valid expansion establishes equivalence

$$3(x + 2) - x$$

$$3x + 6 - x$$

$$2x + 6$$

*Original native-lesson illustration; labels belong to its worked example.*

#### ■ Function machines and reversing operations · Foundation

Follow the operations in their stated order. To recover a starting input, undo the final operation first. A table pairs each input with its output. The same starting input must have only one output for the rule to be a function.

$$x \mapsto 2x + 3$$

Input 4 gives  $2 \times 4 + 3 = 11$ . To find the unknown input for output 11, subtract 3 to get 8 then divide by 2 to get 4. Inputs -1,0,1 give outputs 1,3,5. A table lists each input beside its output; the operations are always performed in the same order.

Reversing the rule does not mean repeating it. The last forward step is the first reverse step. Different operation orders can produce different outputs.

AQA A7 Foundation interprets simple functions as input-output rules. Higher also uses formal inverse and composite function notation in the separate function lesson.

#### ■ Function machines and reversing operations · Higher

Follow the operations in their stated order. To recover a starting input, undo the final operation first. A table pairs each input with its output. The same starting input must have only one output for the rule to be a function.

$$x \mapsto 2x + 3$$

Input 4 gives  $2 \times 4 + 3 = 11$ . To reverse output 11, subtract 3 to get 8 then divide by 2 to get 4. Inputs -1,0,1 give outputs 1,3,5. If a second machine squares its input, passing 4 through the first then the second gives  $11^2 = 121$ ; reversing their order gives  $2 \times 16 + 3 = 35$ .

Reversing the rule does not mean repeating it. The last forward step is the first reverse step. Different operation orders can produce different outputs.

AQA A7 Foundation interprets simple functions as input-output rules. Higher also uses formal inverse and composite function notation in the separate function lesson.

Forward order; reverse by undoing the last step first

Input 4

$\times 2 \rightarrow 8$

$+ 3 \rightarrow 11$

*Original native-lesson illustration; labels belong to its worked example.*

## ■ Domains, inverses and composition · Higher

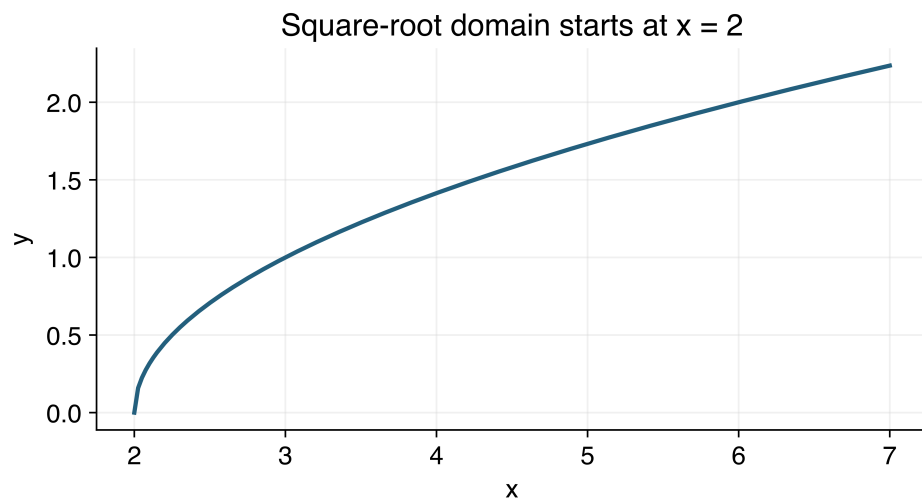
State the domain and range. For an inverse, first ensure the function is one-to-one on its domain. Composition  $fg$  means apply  $g$  first, then  $f$ ; the intermediate output must be an allowed input to  $f$ .

$$f(g(x)) = (f \circ g)(x), \quad f^{-1}(f(x)) = x$$

For  $f(x)=\sqrt{x-2}$ ,  $x \geq 2$  and the range is  $y \geq 0$ . From  $y=\sqrt{x-2}$ ,  $x=y^2+2$ . Thus  $f$  inverse( $x$ )= $x^2+2$  with  $x \geq 0$ . For  $g(x)=x+3$ ,  $fg(1)=f(4)=\sqrt{2}$ .

Squaring can introduce extraneous solutions. Restricting a parabola's domain is essential before claiming an inverse. A horizontal translation inside  $f$  has the opposite sign to the graph's movement.

Check  $f(f \text{ inverse}(x))=x$  on the inverse domain. Use a sketch to test whether a horizontal line meets the original graph more than once.



*Original native-lesson illustration; labels belong to its worked example.*

## ■ Coordinate quadrants and basic graph families · Foundation

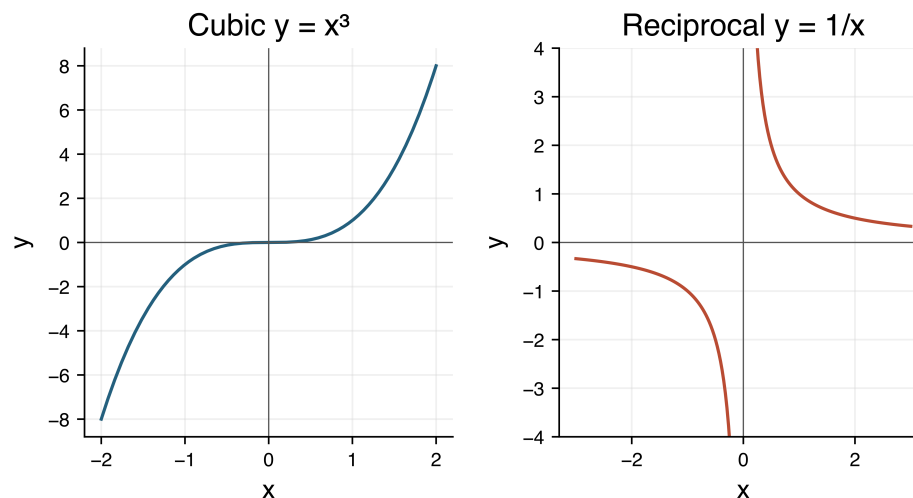
Coordinates are ordered  $(x,y)$ . Signs identify the four quadrants. Use a value table, intercepts and symmetry to sketch a line, quadratic, cubic or reciprocal. For  $y=1/x$ , zero is excluded and the axes are asymptotes. Approximate intersections give graphical solutions.

$$y = x^3, \quad y = \frac{1}{x} \quad (x \neq 0)$$

For  $y=x^3$ , inputs  $-2,-1,0,1,2$  give  $-8,-1,0,1,8$ . For  $y=1/x$ , inputs  $-2,-1,1,2$  give  $-1/2,-1,1,1/2$ ; never substitute zero. The point  $(-2,3)$  is in quadrant II. The graphs  $y=x^2$  and  $y=4$  meet at  $x=-2$  and  $x=2$ . A quadratic  $y=(x-3)(x-7)$  has roots  $3,7$  and symmetry line  $x=5$ .

Join a reciprocal branch smoothly on its own side of zero; never draw a segment through the excluded input. A graph table needs enough points to reveal a turning point or shape.

AQA A8/A11/A12 Foundation includes all quadrants, lines, quadratics, simple cubics and reciprocals. Higher exponential and degree-based trigonometric families are treated separately.



*Original native-lesson illustration; labels belong to its worked example.*

## ■ Exponential and degree-based trigonometric graphs · Higher

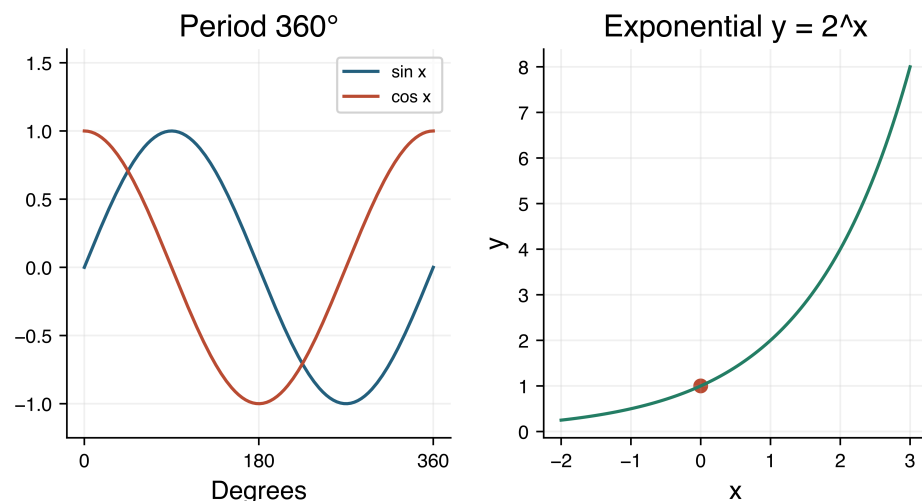
For  $y=k^x$  with  $k>0, k \neq 1$ , the  $y$ -intercept is  $1$ ;  $k>1$  gives growth and  $0<k<1$  gives decay. In degrees, sine and cosine repeat every  $360^\circ$  with range  $[-1,1]$ ; tangent repeats every  $180^\circ$  and is undefined at  $90^\circ+180^\circ n$ . Label axes in degrees.

$$\sin 270^\circ = -1, \quad \tan 45^\circ = 1$$

For  $y=2^x$ ,  $x=-1,0,1,3$  give  $1/2,1,2,8$ . Sine has values  $0,1,0,-1,0$  at  $0^\circ, 90^\circ, 180^\circ, 270^\circ, 360^\circ$ . Cosine starts at  $1$ ;  $\cos 180^\circ = -1$ . Tangent has  $\tan 45^\circ = 1$  but no finite value at  $90^\circ$ . For  $k=1/2$ , increasing  $x$  produces decay, never a negative output.

Degrees and radians are different units. A steep tangent branch is not a finite point at its asymptote. Exponential growth is not repeated addition of a constant amount.

AQA A12 Higher requires exponential functions with positive bases and sine/cosine/tangent for angles of any size in degrees. No differentiation or radian sector formula is introduced here.



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Translations and reflections of function graphs · Higher

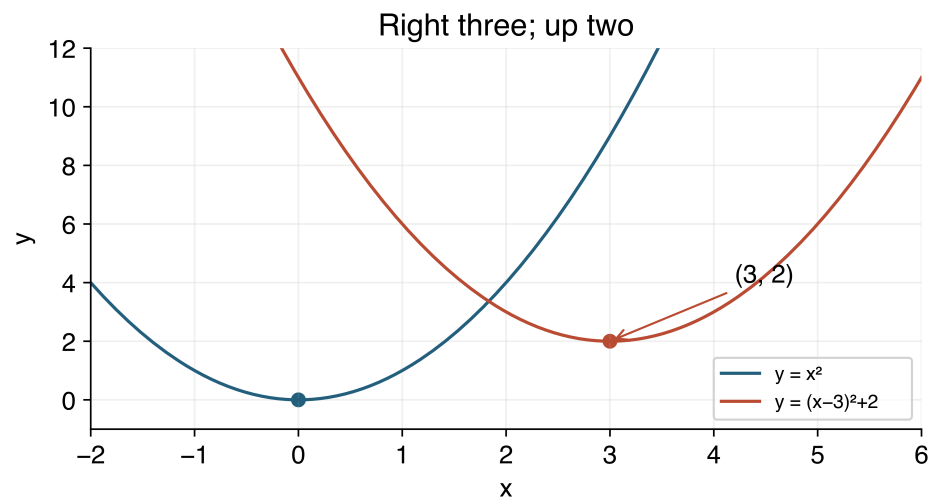
$y=f(x)+a$  shifts the output up by  $a$ .  $y=f(x-a)$  shifts the graph right by  $a$ .  $y=-f(x)$  reflects in the  $x$ -axis;  $y=f(-x)$  reflects in the  $y$ -axis. Transform the points as well as the formula, checking a known feature.

$$y = (x - 3)^2 + 2$$

With  $f(x)=x^2$ ,  $y=f(x-3)+2=(x-3)^2+2$  has vertex (3,2). The point (1,1) on  $f$  moves to (4,3). At  $x=4$  the new output is 3. The reflection  $y=-x^2$  turns the minimum at the origin into a maximum. For a nonsymmetric graph,  $f(-x)$  mirrors each  $x$ -coordinate, not each  $y$ -coordinate.

A positive number added inside the input,  $f(x+3)$ , moves the graph left, not right. A reflection in the  $y$ -axis changes  $x$ ; a reflection in the  $x$ -axis changes  $y$ .

AQA A13 Higher covers translations and reflections of a given function. Use features and transformed points to justify the sketch; do not substitute a geometric enlargement for this graph transformation.



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Distance-time graphs and contextual intersections · Foundation

Read the axis quantities and units before interpreting shape. On a distance-time graph, slope is speed for a segment with increasing distance; a horizontal segment means no distance change. An intersection of two charge graphs gives equal costs. A curved section requires a local rather than one fixed slope.

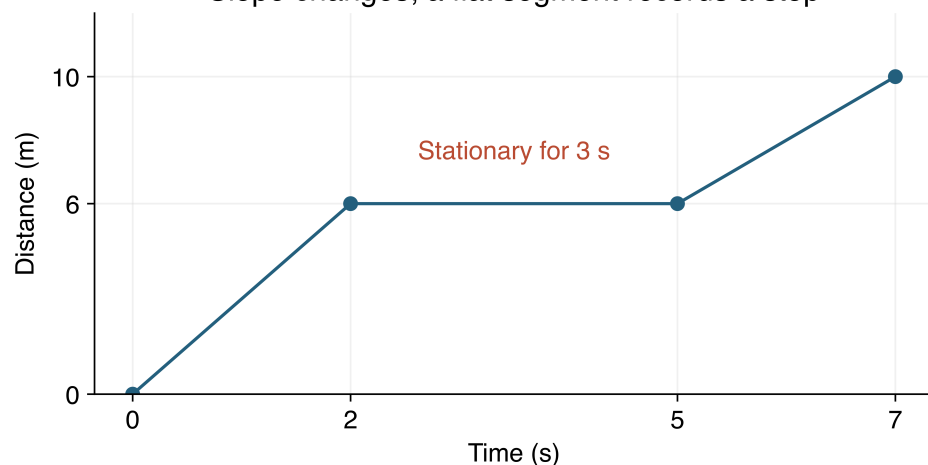
$$v = \frac{\Delta d}{\Delta t}$$

The distance points (0,0),(2,6),(5,6),(7,10), in seconds and metres, give speed  $6/2=3$  m/s for the first section, a 3-second stop, then speed  $(10-6)/(7-5)=2$  m/s. Average speed for the full interval is  $10/7$  m/s, including the stop. Charges  $20+3x$  and  $44+x$  meet at  $x=12$ , at cost 56. If speed increases from 0 to 10 m/s over 5 seconds, its average acceleration is  $10/5=2$  m/s<sup>2</sup>; a speed-time graph measures this through its slope.

A horizontal distance graph does not mean fast motion. A graph height gives distance, while slope gives its rate. Average speed includes every elapsed interval, including waiting.

AQA A14 uses real contexts and graphical solutions. Foundation can interpret a plotted non-standard function; Higher also interprets exponential models and nonlinear graph estimates in the next lesson.

### Slope changes; a flat segment records a stop



Original native-lesson illustration; labels belong to its worked example.

### ■ Graphical gradients and area estimates · Higher

A chord gives an average rate between two inputs; a tangent estimates the local rate. Read two well-separated points on the drawn tangent to reduce measurement error. On a velocity-time graph, area represents displacement. Split it into triangles/trapezia or estimate curved area using narrow strips.

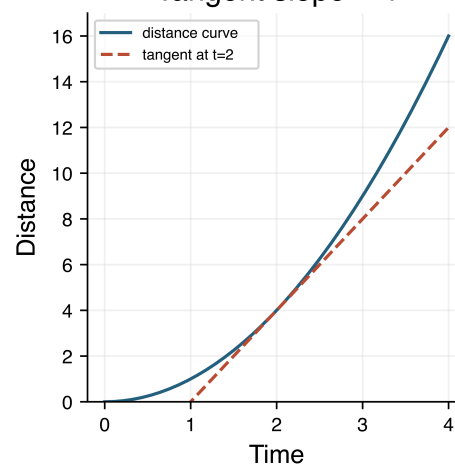
$$\frac{9-1}{3-1} = 4, \quad A = \frac{2+6}{2} \times 3 = 12$$

For a distance curve  $d=t^2$ , the chord from (1,1) to (3,9) has gradient  $(9-1)/(3-1)=4$ . A tangent at (2,4) passes through (1,0) and (3,8), giving gradient 4 without calculus. A velocity-time trapezium with endpoint velocities 2 and 6 m/s over 3 s has area  $(2+6) \times 3/2 = 12$  m. Smaller strips can improve a curved-area estimate; curvature affects over/underestimation. A cost-versus-quantity tangent through (2,12) and (6,28) gives a local rate  $(28-12)/(6-2)=4$  yuan per extra item, linking graph slope to a financial interpretation.

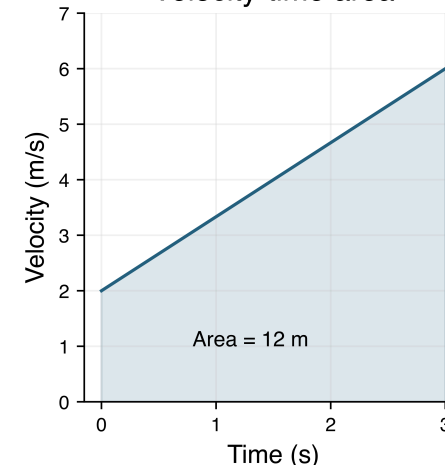
Tangent estimates and curve chords use different pairs of points. Area under a distance-time graph is not distance travelled. A strip estimate is approximate unless the graph is linear on each strip.

AQA A15 and R15 Higher require graph-based rates and area interpretation. This lesson uses graphical reasoning and geometric areas, not symbolic differentiation/integration.

### Tangent slope = 4



### Velocity-time area



Original native-lesson illustration; labels belong to its worked example.

### ■ Origin-centred circles and tangent equations · Higher

For an origin-centred circle,  $x^2+y^2=r^2$ . A point is on the circle if its squared coordinates sum to  $r^2$ . A tangent is perpendicular to the radius there. Use a negative reciprocal gradient when both gradients are finite; handle horizontal/vertical cases directly.

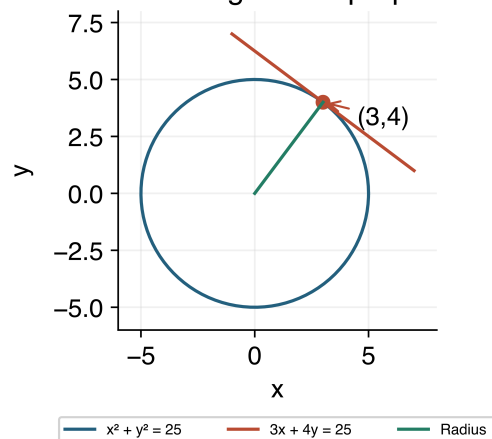
$$x^2 + y^2 = 25, \quad 3x + 4y = 25$$

At (3,4),  $r^2=3^2+4^2=25$ , so  $x^2+y^2=25$ . The radius gradient is  $4/3$ , hence tangent gradient  $-3/4$ . Its equation  $y-4=(-3/4)(x-3)$  simplifies to  $3x+4y=25$ . At (5,0) the radius is horizontal and the tangent is the vertical line  $x=5$ .

The radius and tangent share a point but different directions. Do not use the negative reciprocal of zero; a horizontal radius has a vertical tangent. Check the tangent passes through its contact point.

AQA A16 Higher concerns circles centred at the origin. Offset-centre circle formulae and circle calculus are not needed for this lesson.

Radius and tangent are perpendicular



Original native-lesson illustration; labels belong to its worked example.

## ■ Numerical iteration for equation roots · Higher

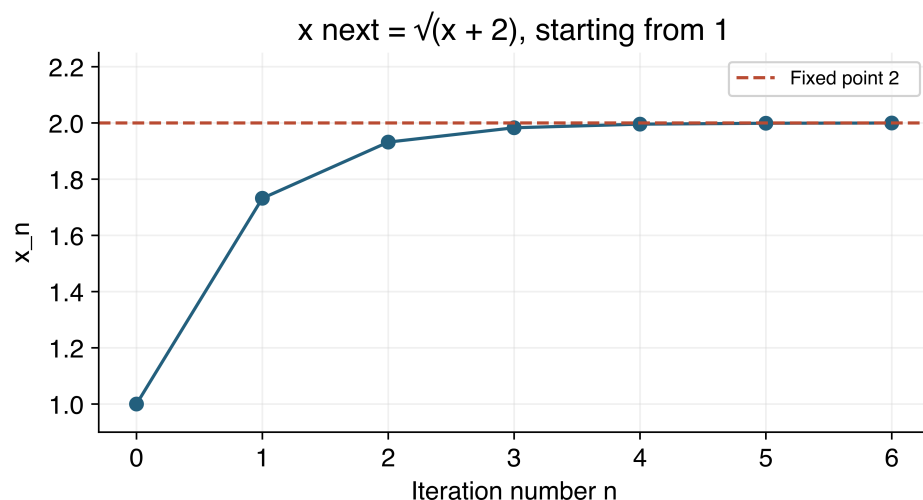
Rearrange an equation as  $x=g(x)$ , choose  $x_0$  and use  $x_{n+1}=g(x_n)$ . Record sufficient working precision. A stable-looking sequence must still be checked in the original equation; not every rearrangement converges. A sign change across continuous inputs can check a rounded root.

$$x_{n+1} = \sqrt{x_n + 2}, \quad x_0 = 1$$

For  $x^2-x-2=0$ , use  $x_{\text{next}}=\sqrt{x+2}$  with  $x_0=1$ :  $x_1=\sqrt{3}\approx 1.73205$ ,  $x_2\approx 1.93185$ ,  $x_3\approx 1.98289$ . The positive fixed point is 2 because  $2=\sqrt{4}$  and  $2^2-2-2=0$ . The rearrangement only seeks a nonnegative root; the original equation also has root -1. Using  $x_{\text{next}}=x^2-2$  from 3 instead gives 7 then 47, so a different rearrangement can diverge.

Use the previous iterate, not the starting value every time. Keep more digits than the final answer. A sign change needs continuity; a jump across a vertical asymptote does not guarantee a root.

AQA A20 Higher uses numerical iteration and suffix notation. Use the calculator to follow the stated rule, then report the requested rounding with a check. Newton derivatives are outside this GCSE lesson.



Original native-lesson illustration; labels belong to its worked example.

## ■ Linear inequalities and number-line solutions · Foundation

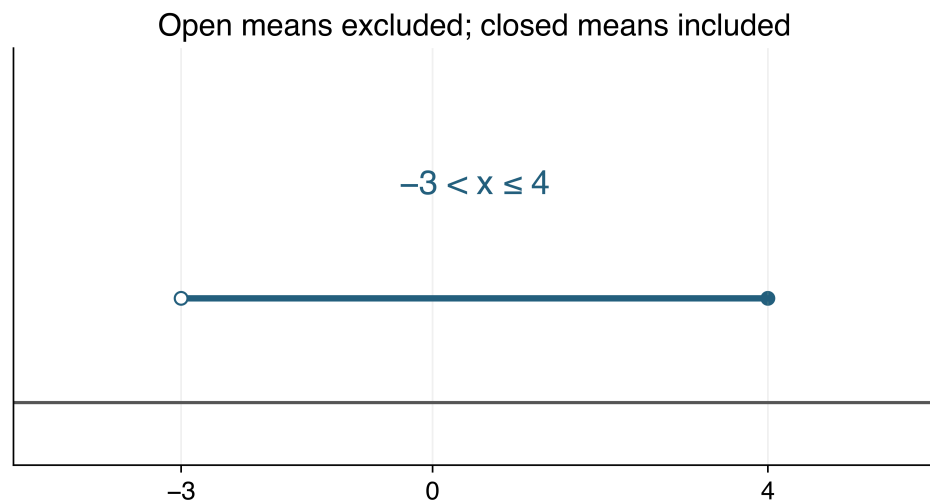
Solve a linear inequality using the same balance operations as an equation. Multiplying or dividing by a negative reverses its direction. Use a closed endpoint for  $\leq$  or  $\geq$  and an open endpoint for  $<$  or  $>$ . Intersect restrictions to find their common permitted inputs.

$$3x + 5 \leq 17 \iff x \leq 4$$

$3x+5\leq 17$  gives  $x\leq 4$ . For nonnegative whole items, the permitted values are 0,1,2,3,4. Also  $-2x<6$  gives  $x>-3$  after division by -2. The combined restriction  $-3<x\leq 4$  has an open circle at -3 and a closed circle at 4. A value  $x=5$  fails the original budget because  $3\times 5+5=20$ .

A reversed sign is needed only when multiplying or dividing by a negative, not when adding a negative. Include the physical domain: negative or fractional item counts may be meaningless.

AQA A22 Foundation requires one-variable linear inequalities and number lines. Higher quadratic and two-variable regions are in a separate lesson; avoid replacing the inequality with a single boundary value.



Original native-lesson illustration; labels belong to its worked example.

### ■ Quadratic inequalities and two-variable regions · Higher

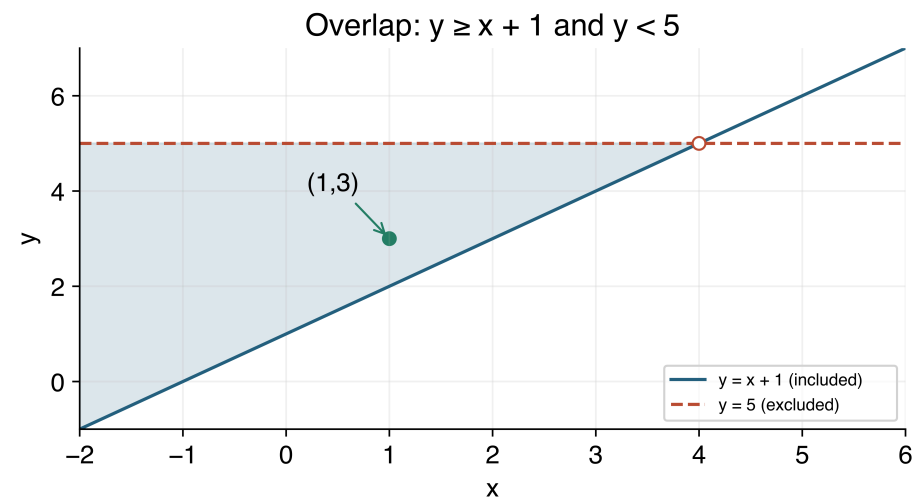
For a quadratic inequality, locate roots and test the sign in each interval. For a two-variable inequality, draw its equality boundary; use a dashed line if equality is excluded and a solid line if included. Test a point on each side and intersect the allowed regions. Write the domain or set notation clearly.

$$(x - 3)(x - 7) \leq 0 \iff x \in [3, 7]$$

$(x-3)(x-7) \leq 0$  holds on  $3 \leq x \leq 7$ , since the factors have opposite signs between the roots and the endpoints give zero. For  $y \geq x+1$  and  $y < 5$ , shade on/above the solid line  $y=x+1$  and below the dashed line  $y=5$ . Their meeting input is  $x=4$ , but  $(4,5)$  is excluded by  $y < 5$ . The point  $(1,3)$  satisfies both inequalities.

For an upward-opening quadratic, positive values lie outside the roots, not between them. A boundary intersection is not necessarily included. Use a test point not on the boundary itself.

AQA A22 Higher includes quadratic inequalities in one variable and linear inequalities in two variables. The graphical overlap is the solution region, not one selected point.



Original native-lesson illustration; labels belong to its worked example.

### ■ Figurate, geometric and Fibonacci-type sequences · Foundation

Record positions and terms separately. Triangular numbers add 1,2,3,...; square and cube numbers use  $n^2$  and  $n^3$ . A geometric sequence multiplies by a common positive ratio. A Fibonacci-type sequence starts with stated terms and then adds its two predecessors. A supplied recursive rule must include enough initial values.

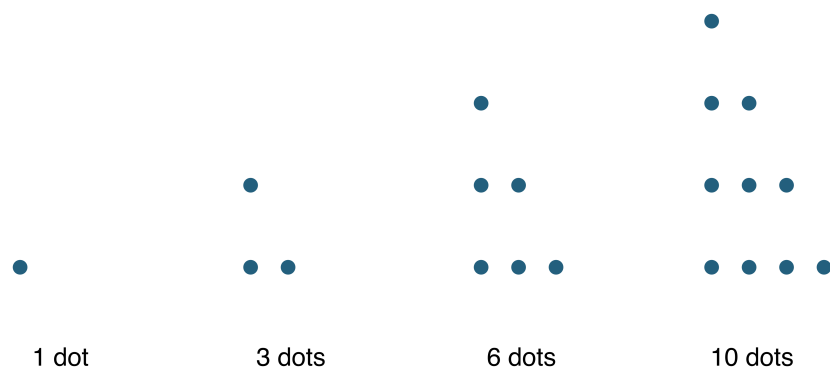
$$u_n = 2 \times 3^{n-1}$$

Triangular terms are 1,3,6,10,15; square terms 1,4,9,16,25; cube terms 1,8,27,64,125. The geometric sequence 2,6,18,54,... has ratio 3 and  $n$ th term  $2 \times 3^{(n-1)}$ . Starting 2,3 and adding the previous two gives 2,3,5,8,13. For  $u_{\text{next}} = 2u + 1$  from  $u_1 = 1$ , the next terms are 3,7,15.

A nonconstant first difference does not mean a pattern is random. A geometric ratio is not a common difference. Check the starting index when writing a position rule.

AQA A23/A24 Foundation includes these patterns and positive rational geometric ratios. Higher also permits surd ratios and derives quadratic  $n$ th terms in the next lesson. Infinite-series sums are not part of this GCSE lesson.

### Triangular patterns add a new row each time



Original native-lesson illustration; labels belong to its worked example.

### ■ Quadratic nth terms and surd-ratio progressions · Higher

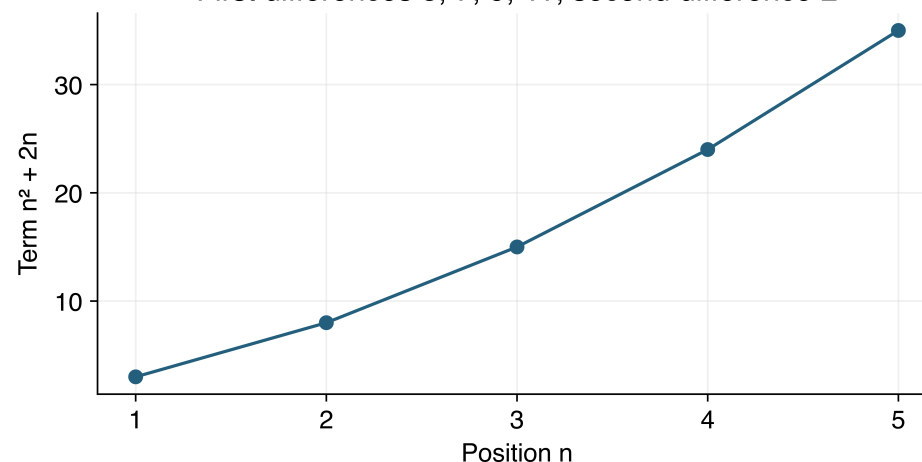
A constant second difference signals a quadratic rule  $an^2 + bn + c$ . Its value is  $2a$ . Subtract  $an^2$  from the terms; fit the remaining linear rule and test at several positions. A geometric sequence with a surd ratio still multiplies by the same exact factor; keep root values exact.

$$u_n = n^2 + 2n, \quad \Delta^2 u_n = 2$$

For 3, 8, 15, 24, first differences are 5, 7, 9 and second difference 2, so  $a = 1$ . Subtract  $n^2$  at  $n=1, 2, 3, 4$  to get 2, 4, 6, 8 =  $2n$ . Thus  $u_n = n^2 + 2n$  and  $u_5 = 35$ . For  $1, \sqrt{2}, 2, 2\sqrt{2}, \dots$  the ratio is  $\sqrt{2}$  and  $u_n = (\sqrt{2})^{(n-1)}$ . Check  $u_3 = 2$  rather than rounding the root repeatedly. The second difference equals  $2a$ , not  $a$ . A quadratic rule must be checked against the first terms; there can be a nonzero constant  $c$ . A finite pattern alone does not prove a unique rule without the stated sequence family.

AQA A25 Higher derives quadratic nth terms; A24 permits surd-ratio sequences. Foundation recognises and generates quadratic patterns without this general derivation.

### First differences 5, 7, 9, 11; second difference 2



Original native-lesson illustration; labels belong to its worked example.

### ■ Algebraic fractions and excluded values · Higher

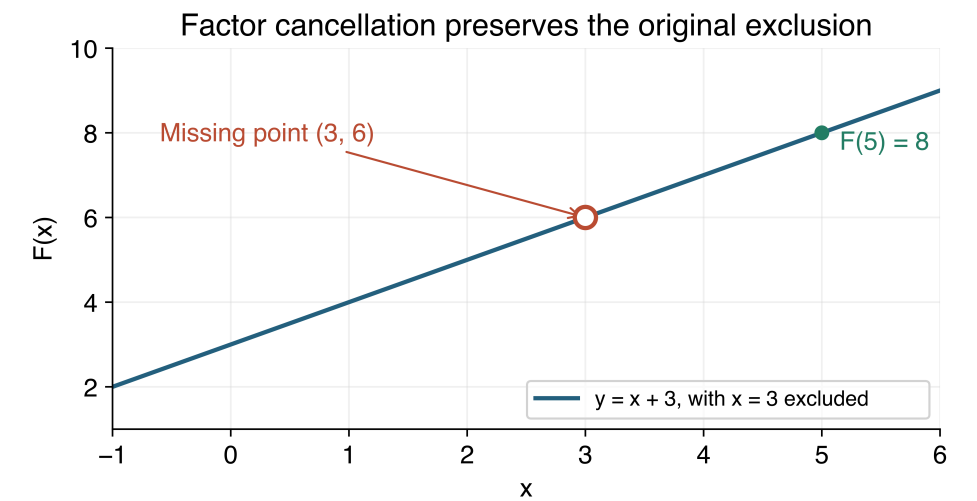
Factor every numerator and denominator before cancelling a common factor. A term joined by addition is not a cancellable factor. Record exclusions from the original expression first. For addition or subtraction use a common denominator, retaining brackets around the entire numerator. Multiply factored numerators and denominators. To divide, multiply by the reciprocal and exclude inputs making the divisor zero as well as inputs making any original fraction undefined.

$$\frac{x^2 - 9}{x - 3} = x + 3 \quad (x \neq 3)$$

For  $F(x) = (x^2 - 9)/(x - 3)$ , factor  $x^2 - 9 = (x - 3)(x + 3)$ . Thus  $F(x) = x + 3$  for  $x \neq 3$ ;  $F(5) = 8$ , but  $F(3)$  is undefined, not 6. For addition,  $1/(x - 1) + 2/(x + 1) = [(x + 1) + 2(x - 1)]/[(x - 1)(x + 1)] = (3x - 1)/(x^2 - 1)$ , with  $x \neq \pm 1$ . At  $x = 3$ , both forms give 1. For subtraction, the numerator is  $(x + 1) - 2(x - 1) = 3 - x$ , so the minus sign acts on both terms. Multiplication  $[(x - 1)/(x + 2)] \times [(x + 2)/(x + 1)]$  gives  $(x - 1)/(x + 1)$ , but the original excludes  $x = -2$  and  $x = -1$ . Division  $[(x^2 - 1)/(x^2 + 3x + 2)] \div [(x - 1)/(x + 2)]$  gives 1, yet  $x = -2, -1$  are excluded:  $x = 1$  makes the divisor zero. Finally  $F(x) = 6$  reduces to  $x + 3 = 6$ . Its only candidate  $x = 3$  is forbidden, so this equation has no solution. Check candidates in the original equation before reporting them.

Never cancel the  $x$  in  $(x + 2)/x$ . A simplified denominator does not show all original restrictions. In a division task, check when the divisor is zero. Cross-multiplication can produce an excluded candidate; a formal root is not automatically a solution.

AQA A4 Higher includes algebraic fractions and valid cancellation. The equation example checks whether a candidate is inside the original domain. Explain each factor cancellation, retain original restrictions and check every equation candidate. These operations extend the existing manipulation example into a complete worked arithmetic sequence.



Original native-lesson illustration; labels belong to its worked example.

### Original independent transfer

■ **Foundation**

A taxi charges a fixed 8 CNY plus 3 CNY per kilometre. Form a cost equation. A trip costs 35 CNY; find its distance and check. Then solve the simultaneous equations  $2x + y = 11$ ,  $x - y = 1$ .

■ **Foundation worked solution**

For distance  $d$  in kilometres and cost  $C$  in CNY,  $C = 8 + 3d$ . Rearranging gives  $d = (C - 8)/3 = (35 - 8)/3 = 9$  km. Substitution gives  $8 + 3(9) = 35$  CNY. Adding the simultaneous equations gives  $3x = 12$ , so  $x=4$  and  $y=3$ . Both equations check:  $2(4) + 3 = 11$ ,  $4 - 3 = 1$ . The cost intercept represents the charge at zero distance, not the distance for zero cost.

■ **Higher**

Solve  $(x^2 - 9)/(x - 3) = x^2 - 5$  on its original real domain. Then solve  $x^2 - 4x - 5 \leq 0$  completely. State the turning point and roots of  $y = x^2 - 4x - 5$ .

■ **Higher worked solution**

The equation excludes  $x=3$ . Cancellation gives  $x + 3 = x^2 - 5$ , hence  $x^2 - x - 8 = 0$  and  $x = (1 \pm \sqrt{33})/2$ . Neither is 3, so both remain. For the separate inequality,  $(x - 5)(x + 1) \leq 0$  gives  $-1 \leq x \leq 5$ , including both zero endpoints. Completing the square gives  $y = (x - 2)^2 - 9$ ; its turning point is  $(2,-9)$ , a minimum, with roots -1 and 5. Do not confuse the two different quadratics.

### Terms

- identity .
- root .
- discriminant .
- elimination .
- gradient .
- common difference .
- common ratio .
- coefficient .
- factorise .
- subject .
- equivalent expression .
- input .
- domain .
- asymptote .
- period .
- graph translation .
- stationary .
- tangent gradient .
- radius gradient .
- iteration .
- solution set .
- boundary line .
- recursive rule .
- second difference .
- excluded value .

# Ratio, proportion and rates of change – native teaching supplement

AQA · GCSE · Mathematics

## Supported teaching and tier boundary

8300: Ratio, proportion and rates of change. Version: Version 1.0, 12 September 2014; first examination 2017.

Foundation teaching and Higher additions are labelled below. This reference packages the existing native-lesson crosswalk. It does not certify unreviewed specification rows or a whole qualification. Original diagnostics are separate and are not reproduced.

### ■ Percentages, ratio and proportional reasoning · Foundation

A  $p\%$  increase has multiplier  $1+p/100$ ; a decrease has multiplier  $1-p/100$ . Reverse a percentage by dividing by the multiplier. In a ratio, first find the total number of parts.

$$P_{\text{new}} = P_{\text{old}} \left( 1 + \frac{r}{100} \right)$$

Let the original price be  $P$ . The model is sale price  $= 0.8P$ . Hence  $P = 240/0.8 = 300$ . A later 20% increase gives  $240 \times 1.2 = 288$ , so the two changes do not cancel.

A percentage uses a stated base. Subtracting the percentages loses that base. For compound change, multiply the multipliers; do not add the percentages.

Use percentage multipliers and divide a total into ratio parts. This Foundation/Core lesson uses linear proportional contexts, not the advanced regression methods.

### ■ Percentages, ratio and proportional reasoning · Higher

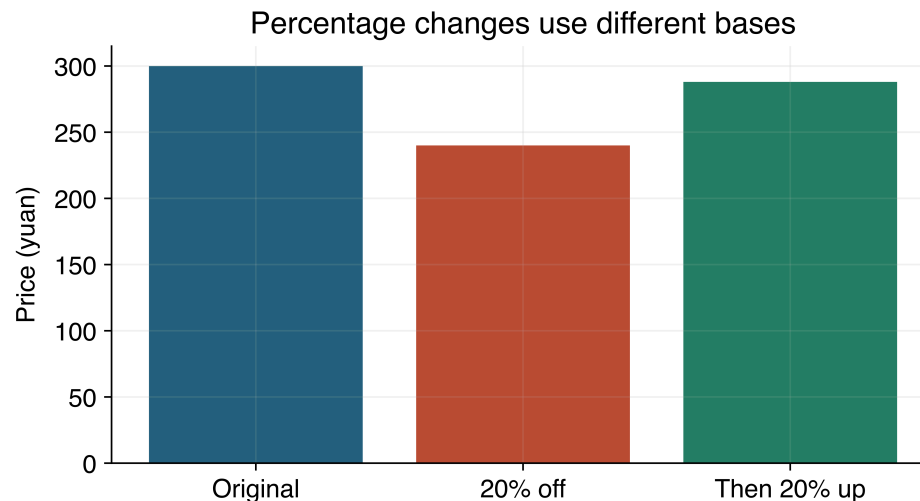
A  $p\%$  increase has multiplier  $1+p/100$ ; a decrease has multiplier  $1-p/100$ . Reverse a percentage by dividing by the multiplier. In a ratio, first find the total number of parts.

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A percentage uses a stated base. Subtracting the percentages loses that base. For compound change, multiply the multipliers; do not add the percentages.

For direct proportion use  $y=kx$ ; for inverse proportion use  $y=k/x$ . Calculate  $k$  from a known pair before using a new value. State what you held constant.



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Ratios, equivalent proportions and mixtures · Foundation

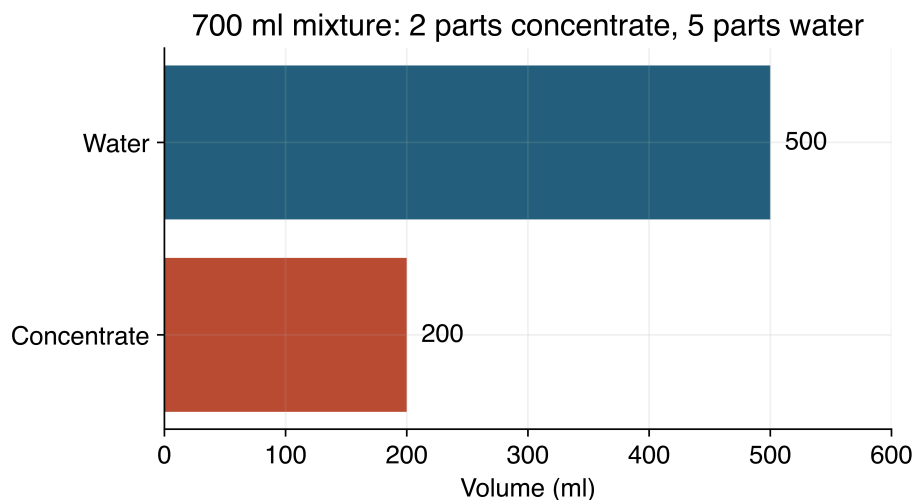
Put quantities in the same units before simplifying a ratio. Divide every part by the same nonzero factor. For  $a:b$  the whole has  $a+b$  parts; the first share is  $a/(a+b)$  of the whole but  $a/b$  of the other share. Equivalent ratios have the same multiplicative relationship:  $a/b=c/d$  gives  $ad=bc$  when denominators are nonzero.

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc, \quad C = \frac{2}{7}V, \quad W = \frac{5}{2}C$$

For 700 ml at 2:5, each part is 100 ml: concentrate 200 ml and water 500 ml. Concentrate:whole  $= 2:7$ , while concentrate/water  $= 2/5$ . Water/concentrate  $= 5/2$ , a fraction greater than 1. Doubling both gives 400:1000  $= 2:5$ . If concentrate is  $x$  and water  $y$ , the recipe requires  $y=(5/2)x$ , a straight line through the origin. For 300 ml concentrate, water is 750 ml and the total is 1050 ml. A batch with 200 ml concentrate and 600 ml water has ratio 1:3 and is weaker. To simplify 1.5 litres:500 ml, first write 1500:500  $= 3:1$ .

Adding the same amount to both shares generally changes the ratio. Do not divide by 2+7 when the stated ratio is already part:whole 2:7; the whole is seven parts. A concentration fraction uses total volume as denominator.

AQA R3–R8 connects ratio notation, fractions, equivalent proportions and linear functions. Check that both shares sum to the stated total, and name which quantity is compared with which. Use a ratio table to scale a mixture without assuming a fixed additive difference.



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Scale drawings and maps · Foundation

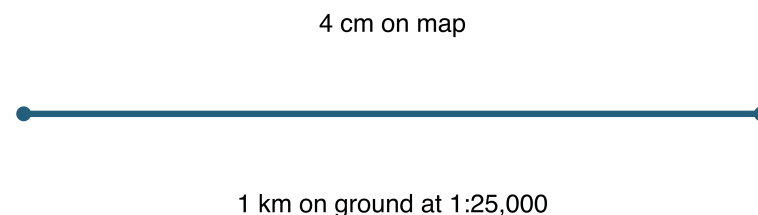
In a scale 1:n, one drawing unit represents n of the same real unit. Multiply a drawing length by n to obtain the real length; divide a real length by n to draw it. Then convert the unit. Measure only when the diagram explicitly supplies an accurate scale.

$$L_{\text{real}} = nL_{\text{drawing}}$$

At 1:25,000, 4 cm represents 100,000 cm=1000 m=1 km. A real 1.5 km path is 150,000 cm, so it measures 150,000/25,000=6 cm on the map. A room 6 m by 4 m drawn at 1:100 becomes 6 cm by 4 cm because each metre is 100 cm. Its drawing diagonal is  $\sqrt{(6^2+4^2)} \approx 7.21$  cm and the real diagonal is about 7.21 m. Enlarging the printed map changes its numerical scale: doubling drawing lengths halves the scale denominator. A scale bar printed with the map enlarges with it.

A ratio compares matching units. Never measure a diagram marked not to scale. A photocopied numerical scale can become invalid even though its scale bar still works. AQA R2 includes maps, scale factors and geometric problems. Label drawing and real dimensions separately; reverse the calculation to verify the scale. Use an exact ratio until the context asks for rounding.

Match units before applying the scale



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Direct and inverse proportional relationships · Foundation

Direct proportion  $y=kx$  preserves  $y/x$  for nonzero  $x$  and gives a straight line through the origin. Inverse proportion  $y=k/x$  preserves  $xy$  and gives a reciprocal curve, with  $x=0$  excluded. State the fixed assumptions: a start-up charge breaks direct proportion, while changing productivity can break inverse proportion. Higher constructs the equation from a known pair; Foundation interprets and uses a given equation.

$$y = kx \quad \text{or} \quad y = \frac{k}{x}, \quad x \neq 0$$

For a given fabric cost  $C=3L$ , lengths 2,4,6 m cost 6,12,18 yuan:  $C/L=3$ . A 5-yuan fixed fee instead gives  $C=5+3L$ , which is linear but not directly proportional. For a given fixed-job model  $t=24/w$ , workers 2,4,6 need 12,6,4 hours, and  $wt=24$  worker-hours. Doubling workers halves time. The inverse model assumes equally productive workers on one fixed job.

An increasing graph need not be direct proportion. An inverse relationship is proportional to  $1/x$ , not to  $-x$ . Keep a product constant for inverse proportion and a quotient constant for direct proportion.

AQA R10/R13/R14 covers numerical, algebraic and graphical direct/inverse proportion. Interpret the units of  $k$  and the gradient. In Foundation use the supplied equations; Higher also constructs and interprets them from the context.

■ Direct and inverse proportional relationships · Higher

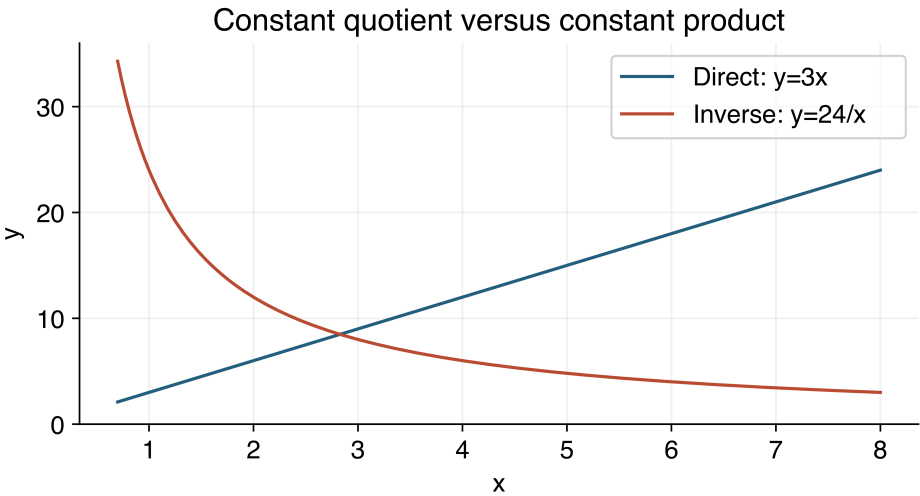
Direct proportion  $y=kx$  preserves  $y/x$  for nonzero  $x$  and gives a straight line through the origin. Inverse proportion  $y=k/x$  preserves  $xy$  and gives a reciprocal curve, with  $x=0$  excluded. State the fixed assumptions: a start-up charge breaks direct proportion, while changing productivity can break inverse proportion. Higher constructs the equation from a known pair; Foundation interprets and uses a given equation.

$$y = kx \quad \text{or} \quad y = \frac{k}{x}, \quad x \neq 0$$

For a given fabric cost  $C=3L$ , lengths 2,4,6 m cost 6,12,18 yuan:  $C/L=3$ . A 5-yuan fixed fee instead gives  $C=5+3L$ , which is linear but not directly proportional. For a given fixed-job model  $t=24/w$ , workers 2,4,6 need 12,6,4 hours, and  $wt=24$  worker-hours. Doubling workers halves time. Higher construction: if  $y$  is directly proportional to  $x$  and  $y=18$  at  $x=6$ ,  $k=18/6=3$ , hence  $y=3x$ . If  $t$  is inversely proportional to  $w$  and  $t=6$  at  $w=4$ ,  $k=tw=24$ , hence  $t=24/w$ . These models assume a constant rate and exclude impossible negative worker counts.

An increasing graph need not be direct proportion. An inverse relationship is proportional to  $1/x$ , not to  $-x$ . Keep a product constant for inverse proportion and a quotient constant for direct proportion.

AQA R10/R13/R14 covers numerical, algebraic and graphical direct/inverse proportion. Interpret the units of  $k$  and the gradient. In Foundation use the supplied equations; Higher also constructs and interprets them from the context.



Original native-lesson illustration; labels belong to its worked example.

■ Rates, unit prices, density and pressure · Foundation

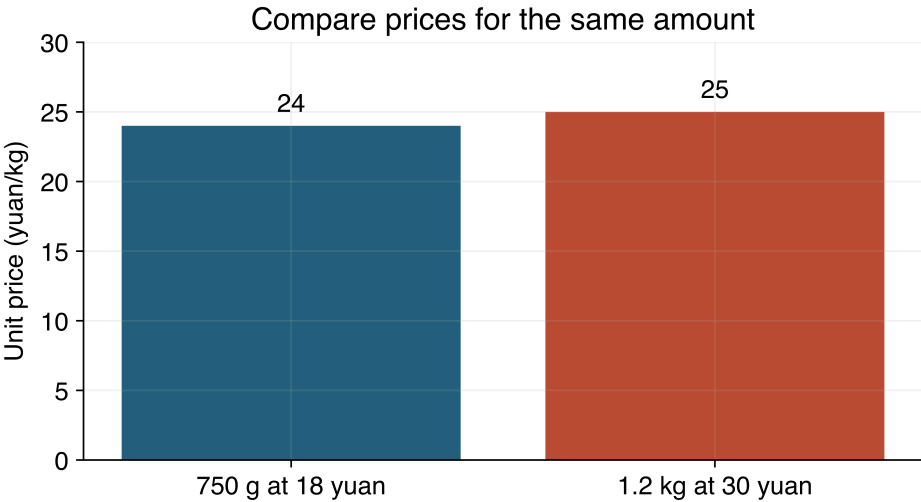
A compound rate divides one quantity by another: speed= $d/t$ , pay=earnings/time, unit price=cost/amount, density=mass/volume and pressure=force/area. Rearrange these equations before substituting. Convert each dimension separately; converting  $\text{cm}^2$  to  $\text{m}^2$  uses a squared length factor. Comparisons must use the same units and conditions.

$$\rho = \frac{m}{V}, \quad p = \frac{F}{A}, \quad m = \rho V$$

A 750 g pack costing 18 yuan has unit price  $18/0.75=24$  yuan/kg. A 1.2 kg pack at 30 yuan costs 25 yuan/kg, so the first is better value if quality and waste are equal. A block of mass 540 g and volume  $200 \text{ cm}^3$  has density  $2.7 \text{ g/cm}^3$ ; a  $50 \text{ cm}^3$  piece of that material has mass 135 g. Since  $1 \text{ g}=0.001 \text{ kg}$  and  $1 \text{ cm}^3=0.000001 \text{ m}^3$ ,  $2.7 \text{ g/cm}^3=2700 \text{ kg/m}^3$ . A 120 N force over  $0.03 \text{ m}^2$  gives pressure  $4000 \text{ N/m}^2=4000 \text{ Pa}$ . At fixed force, halving the area doubles pressure. An hourly pay rate of 48 yuan/hour gives 120 yuan for 2.5 hours.

Volume conversions cube the length factor and area conversions square it. A density of  $2.7 \text{ g/cm}^3$  is not  $2.7 \text{ kg/m}^3$ . A cheaper package may cost more per kilogram; include only comparable products.

AQA R1/R11 includes speed, pay, pricing, density and pressure in numerical and algebraic contexts. Show the rearrangement with named quantities and carry units through the answer; use an inverse check such as density $\times$ volume=mass.



Original native-lesson illustration; labels belong to its worked example.

### ■ Length, area and volume scale factors · Foundation

For similar shapes with corresponding length factor  $k$ , area factor is  $k^2$  and volume factor is  $k^3$ . Name the direction of the comparison: model to real or real to model. Recover  $k$  from an area ratio by a square root and from a volume ratio by a cube root. A change in only one dimension does not create similar solids. Higher links corresponding sides to equal trigonometric ratios.

$$\frac{A_2}{A_1} = k^2, \quad \frac{V_2}{V_1} = k^3$$

Enlarging a box from  $2 \times 3 \times 4$  to  $6 \times 9 \times 12$  multiplies lengths by 3. Its volume changes from 24 to 648, a factor of 27; a  $2 \times 3$  face changes area from 6 to 54, a factor of 9. Length ratio 2:5 corresponds to area ratio 4:25 and volume ratio 8:125. If similar shapes have areas 20 and 80  $\text{cm}^2$ , the larger-to-smaller length factor is  $\sqrt{(80/20)}=2$ . If similar solids have volumes 16 and 128  $\text{cm}^3$ ,  $k=\sqrt[3]{8}=2$ .

Areas do not scale by  $k$ , and volumes do not scale by  $k^2$ . Similarity needs every corresponding length to share the same factor. Reversing a ratio requires the reciprocal factor.

AQA R12 Foundation includes ratios and scale factors for lengths, areas and volumes. Higher adds links to similarity including trigonometric ratios. Check dimensions and compare a simple box or rectangle before applying the general factor.

### ■ Length, area and volume scale factors · Higher

For similar shapes with corresponding length factor  $k$ , area factor is  $k^2$  and volume factor is  $k^3$ . Name the direction of the comparison: model to real or real to model. Recover  $k$  from an area ratio by a square root and from a volume ratio by a cube root. A change in only one dimension does not create similar solids. Higher links corresponding sides to equal trigonometric ratios.

$$\frac{A_2}{A_1} = k^2, \quad \frac{V_2}{V_1} = k^3$$

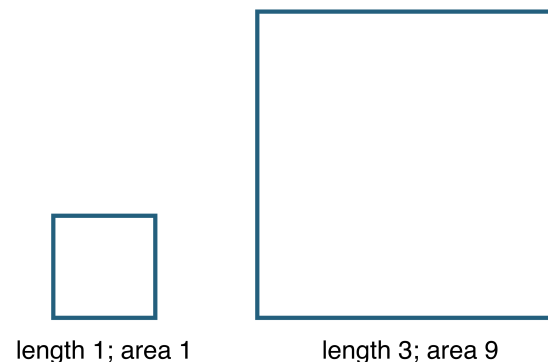
Enlarging a box from  $2 \times 3 \times 4$  to  $6 \times 9 \times 12$  multiplies lengths by 3. Its volume changes from 24 to 648, a factor of 27; a  $2 \times 3$  face changes area from 6 to 54, a factor of 9. Length ratio 2:5 corresponds to area ratio 4:25 and volume ratio 8:125. If similar shapes have areas 20 and 80  $\text{cm}^2$ , the larger-to-smaller length factor is  $\sqrt{(80/20)}=2$ . If similar solids have volumes 16 and 128  $\text{cm}^3$ ,  $k=\sqrt[3]{8}=2$ . Higher: triangles with the same acute angle have equal opposite/hypotenuse ratios; doubling both lengths leaves  $\sin\theta$  unchanged.

Areas do not scale by  $k$ , and volumes do not scale by  $k^2$ . Similarity needs every corresponding length to share the same factor. Reversing a ratio requires the reciprocal factor.

AQA R12 Foundation includes ratios and scale factors for lengths, areas and volumes.

Higher adds links to similarity including trigonometric ratios. Check dimensions and compare a simple box or rectangle before applying the general factor.

### A length factor of 3 acts in both area dimensions



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Simple interest, compound growth and decay · Foundation

A percentage  $r$  corresponds to decimal  $r/100$ ; increases use multiplier  $1+r/100$  and decreases  $1-r/100$ . Simple interest on principal  $P$  for  $n$  periods at rate  $r$  is  $Prn/100$ , giving balance  $P(1+rn/100)$ . Compound balance is  $P(1+r/100)^n$ . Decay uses the corresponding decreasing multiplier. Use matching rate and period units, and divide by the multiplier to recover an original value.

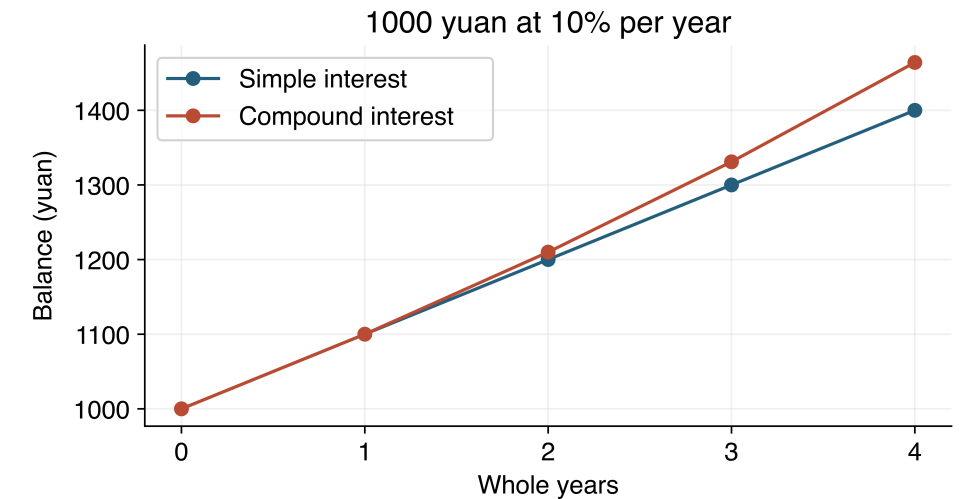
$$B_n = P \left( 1 + \frac{r}{100} \right)^n$$

At 10% per year, 1000 yuan earns simple interest 100 per year: after two years interest is 200 and balance is 1200. Compound balances are 1100 after year one and 1210 after year two. A machine worth 800 yuan losing 20% each year becomes 640 then 512, not 480: the second loss is 20% of 640. An 800-yuan sale price after a 20% reduction corresponds to original price  $800/0.8=1000$ . Growth from 50 to 65 is  $(65-50)/50 \times 100=30\%$ ; 65 is 130% of 50. If a compound balance must first exceed 1300 at 10%, year two gives 1210 and year three 1331, so three whole years are needed.

Use the original value as the denominator for percentage change. Repeated 20% decreases do not subtract 40% of the initial amount. A rate per year cannot be treated as a rate per month without a specified conversion.

AQA R9/R16 includes percentage comparisons, original values, simple interest and

repeated growth/decay. State whether the task asks for interest alone or total balance, and interpret whole-period threshold answers.



Original native-lesson illustration; labels belong to its worked example.

■ Iterative processes with repeated deposits · Higher

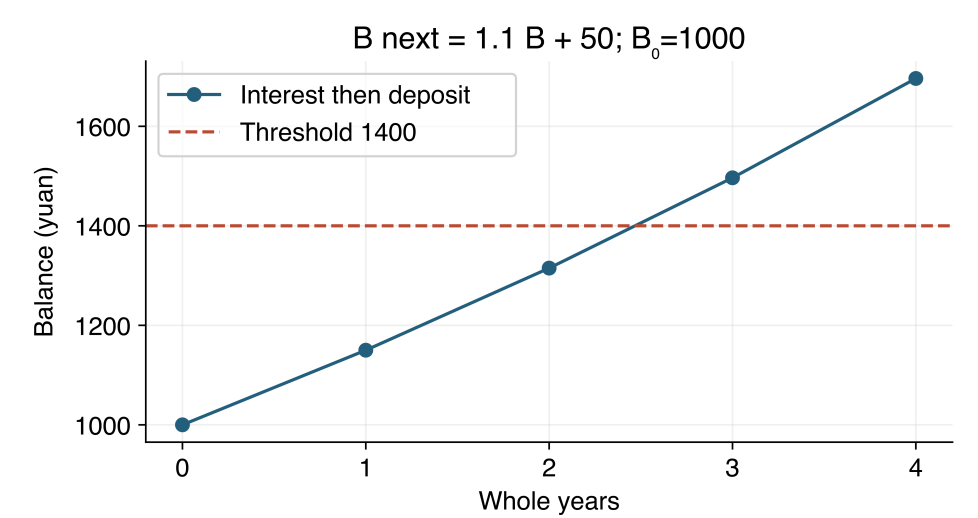
Write an update rule and an initial value. Apply the operations in their stated order, using the previous output as the next input. A recurrence  $B_{\text{next}}=1.1B+50$  differs from  $1.1(B+50)$ . Tables can locate a first whole-period threshold; verify both the preceding and crossing values. Iteration can model growth, decay or other repeated processes, not just solve equations.

$$B_{n+1} = 1.1B_n + 50, \quad B_0 = 1000$$

With  $B_0=1000$  and  $B_{n+1}=1.1B_n+50$ ,  $B_1=1150$ ,  $B_2=1315$  and  $B_3=1496.5$ . The balance first exceeds 1400 after three years because  $1315 \leq 1400 < 1496.5$ . If the deposit preceded interest,  $B_1=1.1 \times 1050=1155$ , five yuan greater. A decay-and-top-up rule  $V_{\text{next}}=0.8V+20$ , starting at 200, gives 180 then 164. A fixed point satisfies  $V=0.8V+20$ , hence  $V=100$ ; values above 100 decrease toward it. A fixed point is a value preserved by the update, not a claim that every finite step reaches it exactly.

Preserve operation order and use the updated value each time. Do not round early or confuse the initial value with the first updated value. A continuous fractional-period estimate does not answer a whole-period question by itself.

AQA R16 Higher extends growth/decay to general iterative processes. State the model assumptions, initial value, recurrence and threshold interpretation. Use substitution to check a proposed fixed point without calculus.



Original native-lesson illustration; labels belong to its worked example.

Original independent transfer

■ Foundation

A 1:25000 map shows a road as 6.4 cm. Find its real length in kilometres. A cyclist travels it in eight minutes. Find average speed in kilometres per hour, showing the unit conversion.

■ Foundation worked solution

Real length  $L = kl = 25000(6.4 \text{ cm}) = 160000 \text{ cm} = 1.6 \text{ km}$ . Time  $t = 8/60 \text{ h} = 2/15 \text{ h}$ . Average speed  $v = L/t = (1.6 \text{ km})/(2/15 \text{ h}) = 12 \text{ km/h}$ . The map ratio compares like units before conversion. The average does not assert constant instantaneous speed.

■ Higher

A savings account starts at 1000 CNY. Each year it receives 5% interest, then a 100 CNY deposit. Write a recurrence and find the balance after two years. Compare with depositing before interest each year, explaining the difference.

■ Higher worked solution

Interest then deposit gives  $B_{n+1} = 1.05B_n + 100$ , with  $B_0 = 1000$ . Thus  $B_1 = 1150$  and  $B_2 = 1307.50$  CNY. Deposit first gives  $C_{n+1} = 1.05(C_n + 100)$ , so  $C_1 = 1155$  and  $C_2 = 1317.75$  CNY. The difference is 10.25 CNY: the first early deposit earns five CNY an extra year and that advantage earns 5% in year two, while the second early

deposit adds another five CNY. Operation order is part of the model.

Terms

- multiplier
- part-to-whole ratio
- scale
- constant of proportionality
- density
- area scale factor
- compound interest
- recurrence relation

Geometry and measures – native teaching supplement

AQA · GCSE · Mathematics

Supported teaching and tier boundary

8300: Geometry and measures. Version: Version 1.0, 12 September 2014; first examination 2017.

Foundation teaching and Higher additions are labelled below. This reference packages the existing native-lesson crosswalk. It does not certify unreviewed specification rows or a whole qualification. Original diagnostics are separate and are not reproduced.

■ Angles, lengths and area · Foundation

Use angle facts with a stated reason. Similar shapes have equal corresponding angles and proportional corresponding lengths. Areas of rectangles and triangles come from their dimensions; compound shapes can be split into simpler parts.

$$A_{\text{rectangle}} = LW, \quad A_{\text{triangle}} = \frac{1}{2}bh$$

A rectangle of length 8 cm and width 5 cm has area  $A=LW=40\text{ cm}^2$ . A triangle on the same base and height has area  $A=bh/2=20\text{ cm}^2$ . For a pentagon, the interior-angle sum is  $(5-2)\times180=540^\circ$ .

Equal angles alone establish similarity, not equal size. Use corresponding lengths in the same order. Convert linear units before calculating area or volume, or square/cube the conversion factor correctly.

Use a labelled sketch and appropriate units. This Foundation/Core lesson does not test area/volume scale factors or advanced circle-theorem proofs.

■ Angle reasoning, similarity and mensuration · Higher

For similar shapes with length scale factor  $k$ , areas scale by  $k^2$  and volumes by  $k^3$ . State angle reasons explicitly. A circle’s tangent is perpendicular to the radius at the contact point.

$$\frac{A_2}{A_1} = k^2, \quad \frac{V_2}{V_1} = k^3$$

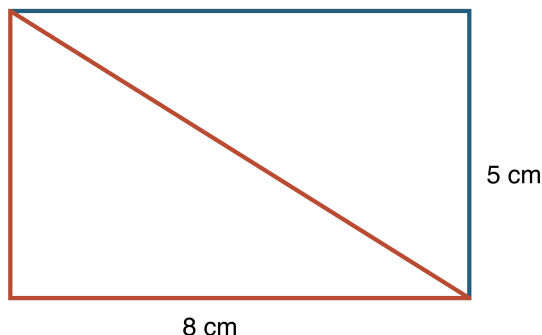
If model-to-real length factor is 3, a model area of  $12\text{ cm}^2$  gives  $12\times3^2=108\text{ cm}^2$  and a model volume of  $8\text{ cm}^3$  gives  $8\times3^3=216\text{ cm}^3$ . A cylinder with  $r=3, h=5$  has volume  $\pi r^2h=45\pi$ .

Equal angles alone establish similarity, not equal size. Use corresponding lengths in the same order. Convert linear units before calculating area or volume, or square/cube the conversion factor correctly.

A geometric proof should name the relevant theorem, identify the equal angle or ratio, and draw the conclusion. A scale drawing is evidence only when the task permits measurement.

### Same base and height, half the area

Rectangle: 40 cm<sup>2</sup>; triangle: 20 cm<sup>2</sup>



*Original native-lesson illustration; labels belong to its worked example.*

### A length scale factor is squared for area



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Right-angled triangles in two dimensions · Foundation

Use Pythagoras in a right triangle and use sine, cosine or tangent with the sides labelled relative to the chosen angle.

$$a^2 + b^2 = c^2, \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

The ladder length is  $c = \sqrt{3^2 + 4^2} = 5$  m. Its angle to the ground satisfies  $\tan \theta = 4/3$ , so  $\theta \approx 53.1^\circ$ . A right triangle with legs 6 and 8 has area  $6 \times 8 / 2 = 24$ .

Label sides relative to the chosen angle. Pythagoras needs a right angle. A calculator angle mode error can produce a plausible but wrong result. Keep unrounded values for later steps.

This Foundation/Core lesson uses right-angled triangles only. Sine and cosine rules for non-right triangles belong to the advanced-tier lesson.

### ■ Right triangles and non-right triangles · Higher

In a right triangle  $a^2 + b^2 = c^2$ ;  $\sin \theta = \text{opposite} / \text{hypotenuse}$ ,  $\cos \theta = \text{adjacent} / \text{hypotenuse}$  and  $\tan \theta = \text{opposite} / \text{adjacent}$ . For other triangles, use the sine or cosine rule, or  $\text{area} = ab \sin C / 2$ .

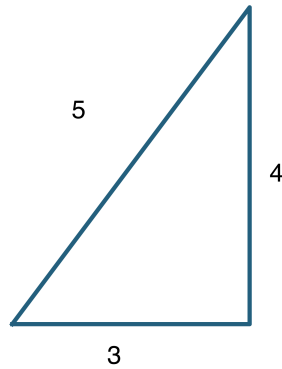
$$a^2 + b^2 = c^2, \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

The ladder length is  $c = \sqrt{3^2 + 4^2} = 5$  m. Its angle to the ground satisfies  $\tan \theta = 4/3$ , so  $\theta \approx 53.1^\circ$ . With two sides 6 and 8 enclosing  $60^\circ$ ,  $c^2 = 6^2 + 8^2 - 2 \times 6 \times 8 \cos 60^\circ = 52$ .

Label sides relative to the chosen angle. Pythagoras needs a right angle. A calculator angle mode error can produce a plausible but wrong result. Keep unrounded values for later steps.

Use a plan or elevation for a three-dimensional problem before applying a triangle rule. Explain why the chosen triangle contains the required length or angle.

Label the known sides before choosing a rule



Original native-lesson illustration; labels belong to its worked example.

#### ■ Vectors and transformation geometry · Foundation

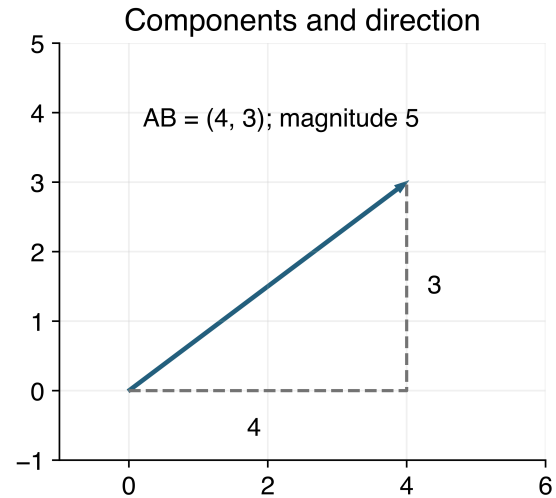
Add corresponding vector components and subtract position vectors to find a displacement. A translation moves every point by the same vector; a scalar multiple changes length and possibly direction.

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

With  $a=(4,1)$  and  $b=(1,3)$ ,  $a+b=(5,4)$  and  $2a-b=(7,-1)$ . From  $A=(1,2)$  to  $B=(5,5)$ ,  $AB=(4,3)$  and its length is 5. Write each 2D vector as a column with the horizontal component above the vertical component: the translation  $AB$  has top entry 4 and bottom entry 3. A negative horizontal entry moves left; a negative vertical entry moves down. Drawing  $b$  from the head of  $a$  gives the head-to-tail diagram for  $a+b$ . Subtracting  $b$  adds its opposite  $-b$ .

The order of subtraction matters:  $BA=-AB$ . Proving parallelism needs a scalar-multiple relation; a sketch alone is insufficient. Negative enlargement reverses position about its centre.

Draw arrows with direction and identify the starting and ending points. Scalar products, spatial line equations and advanced angle calculations are excluded from this Foundation/Core lesson.



Original native-lesson illustration; labels belong to its worked example.

#### ■ Reflections, rotations and enlargements · Foundation

A translation adds a vector. A reflection reverses signed perpendicular distance from a mirror line. A rotation needs a centre, angle and direction. For enlargement from  $C$ , use new  $P=C+k(P-C)$ .

$$\mathbf{p}_{\text{image}} = \mathbf{c} + k(\mathbf{p} - \mathbf{c})$$

For centre  $C=(1,1)$ , point  $P=(3,2)$  and factor 2, the image is  $(5,3)$ . At factor  $1/2$  the image is  $C+(1/2)(P-C)=(2,1.5)$ , halfway from  $C$  to  $P$ . Reflecting  $(3,2)$  in the  $y$ -axis gives  $(-3,2)$ . Rotating it  $90^\circ$  anticlockwise about the origin gives  $(-2,3)$ . Translation by  $(2,-1)$  gives  $(5,1)$ . State the centre, line or vector as appropriate.

Specify a reflection line, rotation centre/angle/direction or enlargement centre/factor. Fractional positive enlargement factors reduce a shape without reversing its position about the centre.

AQA G7 Foundation includes single reflections, rotations, translations and enlargements with positive integer/fractional factors. Negative factors and combinations of isometries belong to separate Higher teaching.

#### ■ Reflections, rotations and enlargements · Higher

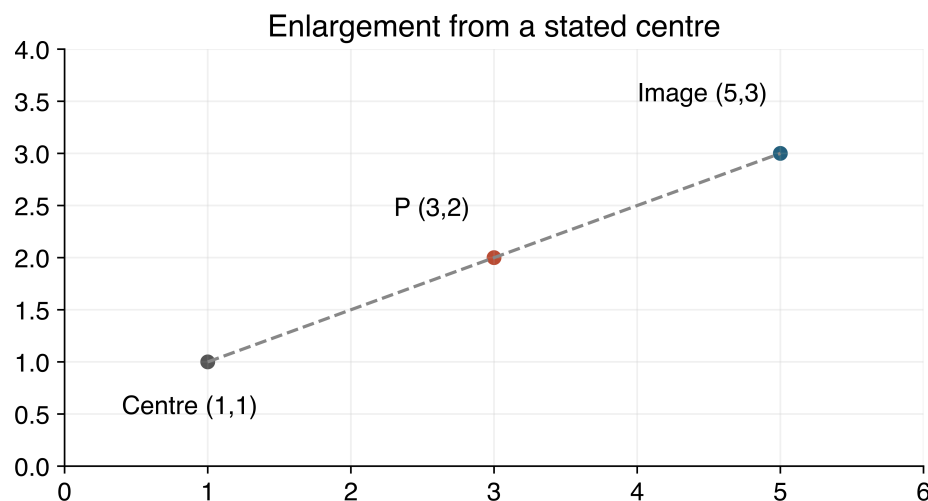
A translation adds a vector. A reflection reverses signed perpendicular distance from a mirror line. A rotation needs a centre, angle and direction. For enlargement from  $C$ , use new  $P=C+k(P-C)$ .

$$\mathbf{p}_{\text{image}} = \mathbf{c} + k(\mathbf{p} - \mathbf{c})$$

For  $C=(1,1), P=(3,2), k=2$ , the image is  $C+2(P-C)=(5,3)$ . With  $k=1/2$  it is  $(2,1.5)$ ; with  $k=-2$  it is  $(-3,-1)$ , on the opposite side of the centre. Reflecting  $(3,2)$  in the  $y$ -axis gives  $(-3,2)$ , and a  $90^\circ$  anticlockwise rotation about the origin gives  $(-2,3)$ . Identify centre, factor, line, angle and direction as applicable.

A rotation needs its centre and direction, not just an angle. A negative enlargement factor places the image on the opposite side of the centre. A translation does not change orientation or size.

Describe a transformation completely before constructing the image. Check corresponding distances and angles. For combined transformations, apply them in the stated order; they usually do not commute.



*Original native-lesson illustration; labels belong to its worked example.*

## ■ Constructions, loci and geometric conditions · Foundation

Points equally distant from A and B lie on the perpendicular bisector of AB. Points at fixed distance  $r$  from C lie on a circle. Points equally distant from two intersecting lines lie on their angle bisectors.

$$PA = PB, \quad h^2 + 3^2 = 5^2$$

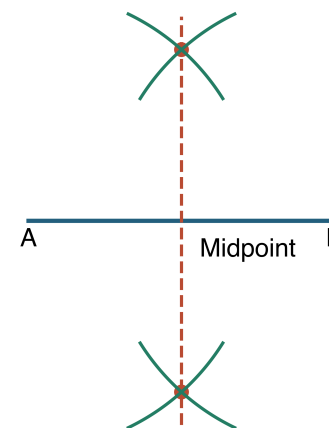
For endpoints A and B 6 cm apart, draw equal-radius arcs above and below AB, with compass opening greater than 3 cm. Join the arc intersections to obtain the perpendicular bisector, crossing AB at its midpoint 3 cm from each end. To construct

a perpendicular from P to a line, use a circle centred at P to mark two line points, then bisect their segment. For a perpendicular at P on the line, mark equal distances on each side of P and use equal arcs. For an angle bisector, draw one vertex-centred arc meeting both arms, then equal arcs from those two points; join their intersection to the vertex. Equal-radius circles centred at the endpoints of a segment construct an equilateral triangle and hence a  $60^\circ$  angle. For a point equally distant from A and B and 5 cm from A, intersect the bisector with a 5 cm circle centred at A. Each intersection is 4 cm perpendicular to AB by a 3–4–5 triangle. The shortest point-to-line distance follows a perpendicular, since any slanted route is a longer hypotenuse.

The perpendicular bisector concerns distance to two points; the angle bisector concerns distance to two lines. A sketch is not a ruler-and-compass construction: preserve arcs as evidence of the method.

AQA G2 uses ruler-and-compass constructions, including a  $60^\circ$  angle, perpendiculars, bisectors and intersections of loci. Preserve construction arcs and justify the equidistance condition.

## Equal compass radii: join the two arc intersections



*Original native-lesson illustration; labels belong to its worked example.*

## ■ Circle theorems and reasoned proofs · Higher

The angle at the centre is twice the angle at the circumference on the same arc. Angles in the same segment are equal. Opposite angles of a cyclic quadrilateral sum to  $180^\circ$ . A radius is perpendicular to a tangent at contact.

$$\theta_{\text{centre}} = 2\theta_{\text{circumference}}$$

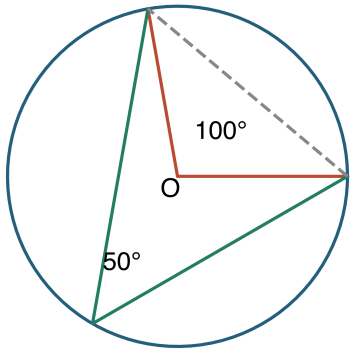
If a central angle is  $100^\circ$ , the corresponding angle at the circumference is  $50^\circ$ . In a

cyclic quadrilateral with one angle  $112^\circ$ , its opposite angle is  $180-112=68^\circ$ . A radius meeting a tangent gives  $90^\circ$ , even if the drawing looks oblique.

Identify the same chord and the correct arc before using a theorem. Two visible right angles do not prove a quadrilateral cyclic without a valid converse argument. A diagram need not be to scale.

Write one reason alongside each angle calculation. For the alternate-segment theorem, name the tangent and chord, then identify the angle in the opposite segment. Use auxiliary radii only when they help the proof.

### The centre angle is twice the circumference angle



*Original native-lesson illustration; labels belong to its worked example.*

#### ■ Shape vocabulary, properties and symmetry · Foundation

A point marks a position; a line extends in both directions and a segment has two endpoints. A plane is a flat two-dimensional surface. A vertex is a corner; an edge is a boundary segment where solid faces meet. A polygon is a closed plane shape made of straight sides; regular means all sides and all interior angles are equal. Points label vertices; AB names a side and angle ABC has vertex B. Parallel lines have the same direction; perpendicular lines meet at  $90^\circ$ . Reflection symmetry uses a mirror line; rotational symmetry counts matches during one full turn, including  $360^\circ$ .

$$\text{order} = \frac{360^\circ}{\text{smallest matching rotation}}$$

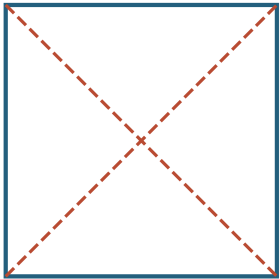
A square has four equal sides, four right angles, four reflection axes and rotational order 4. A non-square rectangle has opposite sides equal, four right angles, two axes and order 2. A non-square rhombus has four equal sides, opposite angles equal, two

diagonal axes and order 2. A parallelogram has two parallel side pairs and opposite angles equal; a general one has no reflection axis. A kite has two pairs of adjacent equal sides; a trapezium has a pair of parallel sides. An equilateral triangle has three equal sides/angles and three reflection axes; an isosceles triangle has two equal sides and equal base angles; scalene means no equal sides. Acute, right and obtuse classify triangles by their largest angle. Pentagons, hexagons, octagons and decagons have 5,6,8,10 sides. A rectangle has equal diagonals that bisect each other. A rhombus has perpendicular bisecting diagonals; a square has both properties. For a parallelogram, a diagonal splits it into triangles: alternate angles on the two parallel side pairs agree and the diagonal is shared, so ASA establishes congruence and opposite sides are equal. Opposite-angle equality also follows from parallel-line angle facts.

Equal-looking lengths need stated equal-length marks or a deduction. Regular does not mean equal sides alone. An axis of symmetry is a full line, not just an internal diagonal.

AQA G1/G4 requires conventional names, notation and derived shape properties. State the property that justifies a classification and allow overlapping classes, such as square and rectangle.

### Square diagonals are symmetry axes



Square



Rectangle

*Original native-lesson illustration; labels belong to its worked example.*

#### ■ Angle facts and polygon reasoning · Foundation

Angles around one point sum to  $360^\circ$ ; angles on a straight line sum to  $180^\circ$ ; vertically opposite angles are equal. When the crossed lines are parallel, corresponding and alternate angles are equal and co-interior angles sum to  $180^\circ$ . A triangle has angle sum  $180^\circ$ . Split an n-sided polygon into n-2 triangles to derive its interior sum (n-

$2) \times 180^\circ$ .

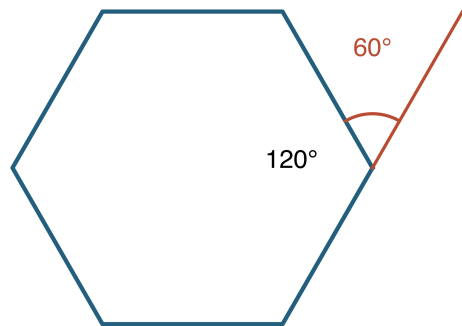
$$S_n = (n - 2)180^\circ, \quad E_{\text{regular}} = \frac{360^\circ}{n}$$

If a straight-line angle is  $68^\circ$ , its neighbour is  $112^\circ$ ; its vertically opposite angle is  $68^\circ$ . A parallel-line corresponding angle is also  $68^\circ$ , with the reason named. Drawing a line through a triangle vertex parallel to the opposite side transfers its two base angles by alternate-angle equality; the three angles then form a straight line and sum to  $180^\circ$ . A pentagon splits into three triangles, giving  $540^\circ$ . A regular hexagon has total  $720^\circ$ , each interior angle  $120^\circ$  and each exterior turn  $60^\circ$ . Exterior turns of a convex polygon sum to one full turn,  $360^\circ$ ; a regular polygon with turn  $45^\circ$  has eight sides.

Corresponding/alternate equalities require parallel lines. Name the theorem rather than using informal letter-shape labels. An interior angle is not the same as an exterior turning angle.

AQA G3/G6 expects reasons and derivations. Mark the given parallelism, identify the angle positions and build a chain of justified equalities rather than reading angles from a sketch.

Regular hexagon: interior  $120^\circ$ , exterior  $60^\circ$



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Triangle congruence and geometric proof · Foundation

Use SSS (three sides), SAS (two sides and their included angle), ASA (two angles and the corresponding side), or RHS (right angle, hypotenuse and one other side). Match vertices in the same order. AAA establishes similarity, not congruence. SSA generally permits more than one triangle. A proof needs a given fact, a valid criterion and a matching-part conclusion.

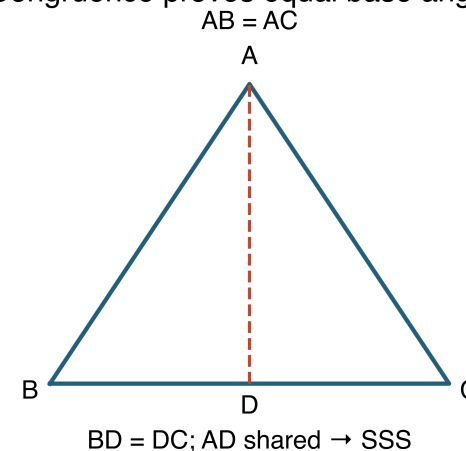
$$(a + b)^2 = 4\frac{ab}{2} + c^2 \implies a^2 + b^2 = c^2$$

For an isosceles triangle ABC with  $AB=AC$ , let D be the midpoint of BC. Triangles ABD and ACD have  $AB=AC$ ,  $BD=DC$  and shared AD, so SSS gives congruence. Matching base angles ABC and BCA are therefore equal; the two angles at D are equal and form  $180^\circ$ , so each is  $90^\circ$ . To derive Pythagoras, arrange four congruent right triangles with legs a,b around a tilted square of side c inside a square of side  $a+b$ . Area gives  $(a+b)^2=4(ab/2)+c^2$ , hence  $a^2+b^2=c^2$ . For  $a=3, b=4$ ,  $c^2=9+16=25$ , so  $c=5$ . These arguments establish results independently of a scale drawing.

The SAS angle must lie between the named sides. RHS uses the hypotenuse, not two arbitrary sides. A proof diagram supports the argument; it cannot establish equality just by appearance.

AQA G5/G6 uses basic congruence criteria and simple geometric proofs, including isosceles base angles and Pythagoras. State every matching pair and explain which criterion applies.

Congruence proves equal base angles



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Circle parts and geometric definitions · Foundation

A circle consists of points a fixed radius from its centre. A diameter is a chord through the centre and has length  $2r$ . A chord joins two circumference points; an arc is part of the circumference. A sector lies between two radii and an arc. A segment lies between a chord and an arc. A tangent touches at one point and is perpendicular to the radius there.

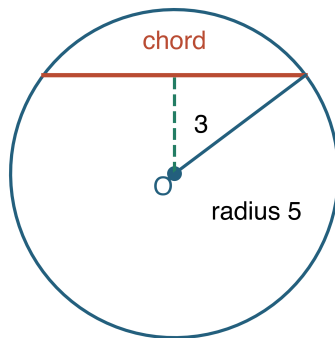
$$d = 2r, \quad \left(\frac{c}{2}\right)^2 + h^2 = r^2$$

For centre O and radius 5 cm, every circumference point is 5 cm from O and the diameter is 10 cm. A chord 3 cm from O has half-length  $\sqrt{(5^2-3^2)}=4$  cm, so its whole length is 8 cm. The perpendicular from O meets the chord at its midpoint. Joining the two chord endpoints to O creates a sector; the smaller region between chord and arc is a segment. At the rightmost circumference point, the vertical touching line is tangent and the horizontal radius is perpendicular to it. The circumference is a length,  $2\pi r=10\pi$  cm, rather than an area.

A chord need not pass through the centre; only a diameter must. Sector and segment have different straight boundaries. Do not confuse circumference length with the shaded area inside a circle.

AQA G9 includes all named circle parts at Foundation. Use the given radius and position labels to identify the part; Higher circle-theorem proofs are developed separately.

### Circle of radius 5: distances use one consistent scale



Perpendicular bisects chord: half-length 4

*Original native-lesson illustration; labels belong to its worked example.*

### ■ Geometric reasoning on coordinate axes · Foundation

Find horizontal or vertical length by subtracting coordinates; use Pythagoras for a diagonal. Midpoint coordinates are averages of the endpoints. Equal coordinate changes identify translations. To prove a quadrilateral property, justify side directions and lengths, rather than naming a shape from its appearance.

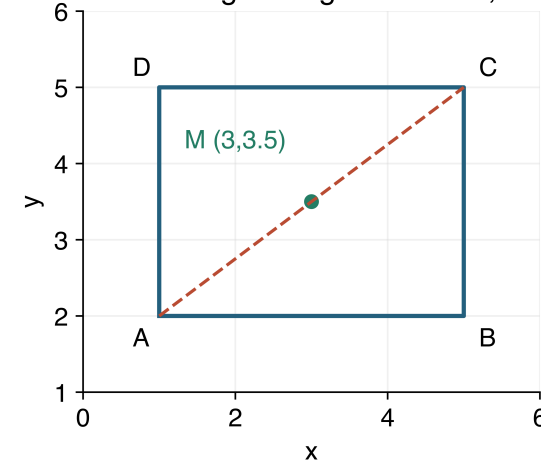
$$M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

For A=(1,2), B=(5,2), C=(5,5), D=(1,5), AB=4 and BC=3. AB is horizontal, BC vertical, so they are perpendicular; opposite sides have matching directions and lengths, establishing a rectangle. Diagonal AC has length  $\sqrt{(4^2+3^2)}=5$ . Its midpoint is  $((1+5)/2, (2+5)/2)=(3,3.5)$ . Diagonal BD has the same midpoint, confirming that the diagonals bisect each other. A translation by (2,-1) sends A to (3,1) and C to (7,4), preserving the diagonal length. A square would additionally need adjacent side lengths equal; these lengths 4 and 3 exclude it.

A coordinate difference can be negative even though a length is nonnegative. The diagonal is not the sum of the two edge lengths. One pair of equal sides alone does not establish a rectangle.

AQA G11 solves geometric problems on axes. Keep point labels, coordinate arithmetic and geometric reasons linked in the written argument; use the diagram to choose a method, then verify it numerically.

### Coordinate rectangle: lengths 4 and 3; diagonal 5



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Solid properties, plans and elevations · Foundation

A face is a flat boundary polygon; curved surfaces must be named separately. Edges join faces and vertices are corners. A prism has two congruent parallel end faces and a constant cross-section. A pyramid joins one polygon base to an apex. Plan is the view from above; front and side elevations are direct views without perspective. Give the viewing direction and align corresponding widths, depths and heights.

$$A_{\text{plan}} = wd, \quad A_{\text{front}} = wh$$

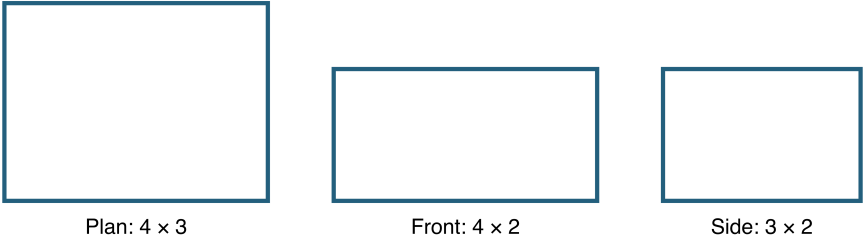
A cube or cuboid has 6 faces, 12 edges and 8 vertices. A triangular prism has 5 faces,

9 edges and 6 vertices; a square pyramid has 5 faces, 8 edges and 5 vertices. A cylinder has two circular flat faces and one curved surface, with no vertices. A cone has one flat circular face, one curved surface and one apex; a sphere has only a curved surface. For a cuboid of width 4, depth 3 and height 2 units, the plan is 4 by 3, front elevation 4 by 2 and side elevation 3 by 2. A stack with front-row column heights 1 and 3, and back-row heights 2 and 1, occupies four cells. The front elevation has column maxima 2 and 3; the side elevation has depth-row maxima 3 and 2. These views do not determine every hidden cube uniquely.

Do not draw perspective diagonals in an orthographic elevation. A plan alone cannot give height. Different hidden arrangements can share the same plan and elevations; do not claim a unique reconstruction without enough information.

AQA G12/G13 covers the named solids and construction/interpretation of views. Label dimensions and directions, preserve alignment between views and explain any hidden-space ambiguity.

Cuboid width 4, depth 3, height 2



Original native-lesson illustration; labels belong to its worked example.

■ Measuring angles and three-figure bearings · Foundation

Draw a north line at the starting point, then measure clockwise to the route. Write three digits: east 090°, south 180°, west 270°, north 000°. NE, SE, SW and NW are 045°,135°,225°,315°. For the reverse bearing add 180° and reduce modulo 360°. Use a ruler for a stated-scale length and the correct protractor scale for a stated angle.

$$b_{\text{reverse}} = (b + 180^\circ) \bmod 360^\circ$$

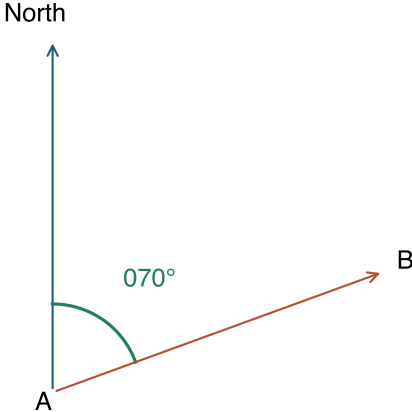
From A to B the bearing 070° is 70° clockwise from north. From B to A it is

070+180=250°. A bearing of 320° reverses to 500-360=140°. East is 090°, not 90 without its three-digit form. A route bearing 120° points southeast of the starting point, making 30° below east. On a 1:10000 map, a 3 cm route represents 300 m; to construct bearing 120°, place the protractor centre at the route start, align its zero with north and measure clockwise. Keep the ruler scale and angular direction as separate decisions.

A bearing is measured at the departure point, not the destination. Read clockwise from north rather than the acute angle to the nearest compass axis. The back bearing differs by 180°, even when the diagram is oblique.

AQA G15 includes measured lengths/angles, maps, the eight compass directions and three-figure bearings. State which point supplies north and distinguish a numerical bearing from a measured scale distance.

Clockwise from north at A; reverse bearing at B is 250°



Original native-lesson illustration; labels belong to its worked example.

■ Combined isometries and invariants · Higher

Apply each transformation to the current image in the stated order. Rotations, reflections and translations preserve lengths and angles, so their combinations also preserve them. Reflections reverse orientation; rotations and translations preserve it. Two reflections in parallel lines give a translation; in intersecting lines they give a rotation through twice the directed angle between the mirrors.

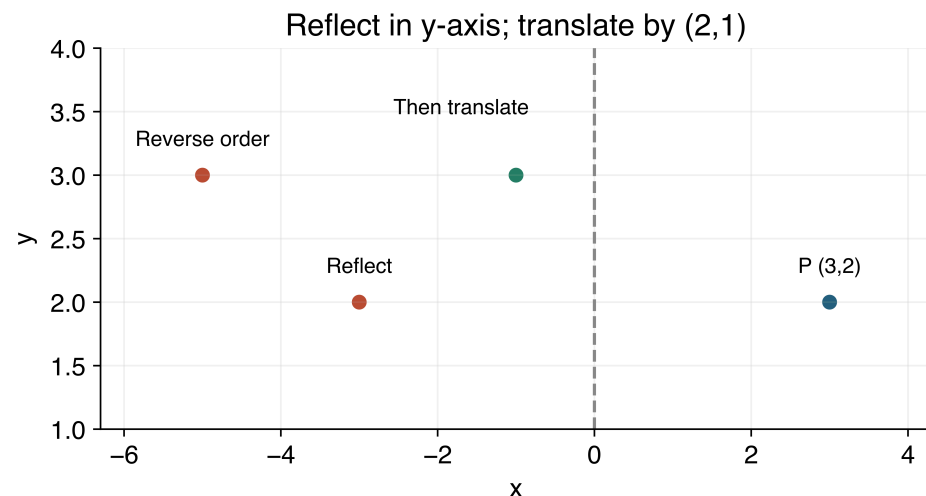
$$T \circ R \neq R \circ T \quad \text{in general}$$

Start with P=(3,2). Reflect in the y-axis to get (-3,2), then translate by (2,1) to get (-1,3). Reversing the order gives (5,3) then (-5,3), a different point. Reflecting in x=0

followed by  $x=2$  maps  $(3,2)$  to  $(-3,2)$  then  $(7,2)$ , equivalent to translation by  $(4,0)$ . Two reflections reverse orientation twice, restoring it. Reflecting in the  $x$ -axis then  $y$ -axis sends  $(3,2)$  to  $(-3,-2)$ , a  $180^\circ$  rotation about the origin. Every pairwise length and angle is unchanged, but position generally changes.

Preserved length does not mean every point stays fixed. Do not commute transformations unless a checked argument permits it. A single reflection reverses orientation, while two reflections restore it.

AQA G8 Higher requires changes and invariance under combinations of rigid transformations. Describe the resulting map completely and test it on more than one point before making a whole-shape claim.



*Original native-lesson illustration; labels belong to its worked example.*

## ■ Circle angles and proof chains · Higher

The central angle on an arc is twice a circumference angle standing on that same arc. A diameter therefore gives a  $90^\circ$  circumference angle. Angles in the same segment are equal. Opposite angles of a cyclic quadrilateral sum to  $180^\circ$ . State the chord, arc and segment before applying a theorem.

$$\angle AOB = 2\angle ACB, \quad \angle A + \angle C = 180^\circ$$

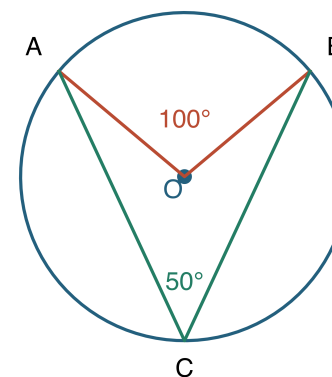
Let  $A, B, C$  lie on the circle and let  $O$  be the centre inside angle  $ACB$ . Put  $\alpha = \text{angle } ACO$  and  $\beta = \text{angle } OCB$ . Equal radii make triangles  $AOC$  and  $BOC$  isosceles. Thus angle  $AOC = 180 - 2\alpha$  and angle  $COB = 180 - 2\beta$ . Angles around  $O$  give the remaining angle  $AOB = 360 - (180 - 2\alpha) - (180 - 2\beta) = 2(\alpha + \beta) = 2 \text{ angle } ACB$ . Other centre positions need the corresponding subtraction of isosceles angles, with the same result for the chosen arc.

If  $AB$  is a diameter, angle  $AOB = 180^\circ$ , so angle  $ACB = 90^\circ$ . Two circumference angles on the same chord in the same segment each equal half the same central angle, so they agree. For opposite cyclic angles, their arcs together make  $360^\circ$ ; half-arc angles therefore sum to  $180^\circ$ . A central angle  $100^\circ$  gives  $50^\circ$  at the circumference; an angle opposite  $112^\circ$  in a cyclic quadrilateral is  $68^\circ$ .

Distinguish the reflex central angle from the smaller one and identify which arc does not contain the circumference vertex. Opposite segments can give supplementary rather than equal angles. A theorem must be tied to named points.

AQA G10 Higher requires application and proof. Draw auxiliary radii, use isosceles base angles and point sums, then extend the proof to the intended configuration rather than inferring equality from the drawing.

## Angles on the same upper arc: centre = twice circumference



*Original native-lesson illustration; labels belong to its worked example.*

## ■ Tangents, chords and the alternate segment · Higher

A tangent is perpendicular to its contact radius. Tangents from one external point are equal. The perpendicular from the centre to a chord bisects it. The tangent–chord angle equals the angle on that chord in the alternate segment. Use equal radii, right triangles and the central-angle theorem to prove these relationships.

$$PA = PB, \quad OM \perp AB \implies AM = MB$$

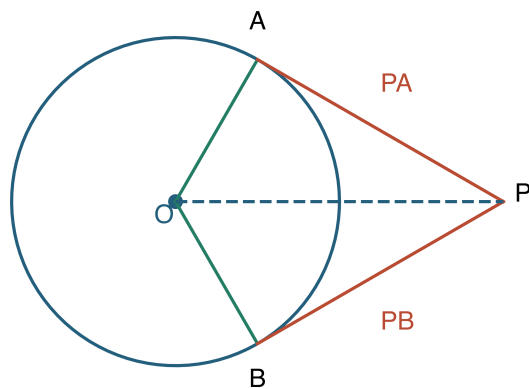
At contact  $T$ , the radius  $OT$  is perpendicular to the tangent: a non-perpendicular line through  $T$  would have a smaller centre-to-line distance than the radius and cut the circle twice. For tangents  $PA$  and  $PB$ ,  $OA = OB$ ,  $OP$  is shared and both contact angles are  $90^\circ$ . RHS makes  $OAP$  and  $OBP$  congruent, so  $PA = PB$ . For a centre perpendicular  $OM$

to chord AB,  $OA=OB$  and  $OM$  is shared; RHS gives  $AM=MB$ . If  $OA=5$  and  $OM=3$ ,  $AM=4$ , hence  $AB=8$ . For a chord AB with minor central angle  $\varphi$ , triangle OAB has base angle  $(180-\varphi)/2$ . The adjacent tangent–chord angle is  $90-(180-\varphi)/2=\varphi/2$ , equal to the circumference angle in the alternate segment. Thus a tangent–chord angle of  $35^\circ$  gives  $35^\circ$  in that segment. Reflex/supplementary configurations require the matching arc and angle.

A tangent–chord angle and a radius–chord angle are different. Equal tangent lengths refer to one external point. The chord is bisected by a perpendicular from the centre, not by any line that happens to cross it.

AQA G10 Higher includes these theorem proofs and related results. Give the congruence criterion or angle chain explicitly and identify the relevant chord, contact point and alternate segment.

Shared OP and equal radii  $\rightarrow$  RHS  $\rightarrow PA = PB$



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Areas, perimeters and composite plane shapes · Foundation

Triangle area is  $bh/2$ , parallelogram area  $bh$  and trapezium area  $(a+b)h/2$  for parallel sides  $a, b$ . Heights are perpendicular to the selected base. Perimeter adds only the outside boundary. Split a composite shape into non-overlapping parts, or subtract a missing region from a containing shape.

$$A_{\triangle} = \frac{1}{2}bh, \quad A_{\text{trap}} = \frac{1}{2}(a+b)h$$

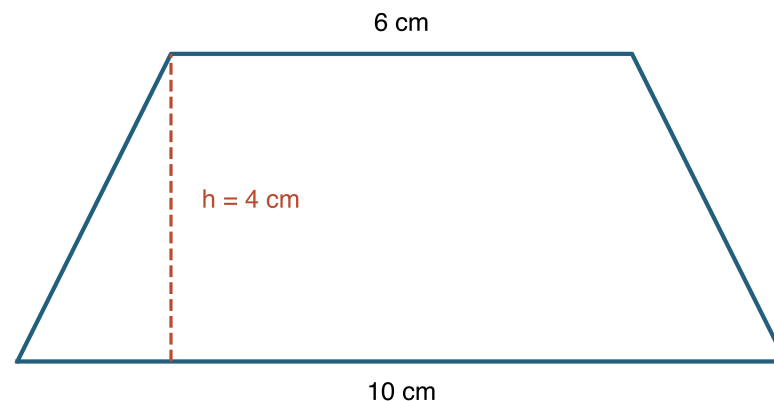
A triangle with base 8 cm and height 5 cm has area  $20 \text{ cm}^2$ ; a parallelogram with the same base/height has area  $40 \text{ cm}^2$ . A trapezium with parallel sides 6 and 10 cm and

height 4 cm has area  $(6+10) \times 4/2 = 32 \text{ cm}^2$ . A rectangular 8 by 6 cm sheet with a 3 by 2 cm corner removed has area  $48-6=42 \text{ cm}^2$ . Its perimeter is still 28 cm: the two removed outside segments total 5 cm and the two new notch edges also total 5 cm. This perimeter equality depends on a corner rectangular cut; an internal hole adds a separate boundary. Rearranging triangle area gives  $h=2A/b$ .

Use perpendicular height rather than a sloping side. A shared internal division line is not part of the perimeter. An area answer has squared units; a perimeter answer has length units.

AQA G16/G17 includes triangle/parallelogram/trapezium area and composite perimeter/area. Mark the parallel bases, perpendicular height and counted outside edges before calculating.

Trapezium area = average parallel sides  $\times$  height



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Prism volume and cylinder measurement · Foundation

For a right prism, volume is constant cross-sectional area times perpendicular length. Cuboid volume is  $lwh$ . A cylinder is a circular prism:  $V=\pi r^2h$ . Total closed-cylinder surface area includes two circular ends and the curved surface  $2\pi rh$ . Open containers omit the specified faces.

$$V = A_{\text{cross}}L, \quad V_{\text{cylinder}} = \pi r^2h$$

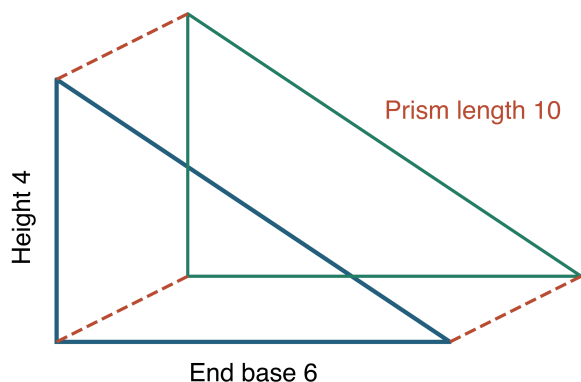
A triangular prism with end base 6 cm, end height 4 cm and length 10 cm has cross-section  $12 \text{ cm}^2$  and volume  $120 \text{ cm}^3$ . A cylinder of radius 3 cm and height 5 cm has volume  $45\pi \text{ cm}^3$ . Its curved surface unwraps to a rectangle of width  $2\pi r=6\pi$  and height 5, so curved area is  $30\pi \text{ cm}^2$ . Adding two ends gives total area  $30\pi+18\pi=48\pi$

cm<sup>2</sup>. An open-top tank of those dimensions has surface area  $39\pi$  cm<sup>2</sup> because it keeps only one end. If a prism has volume 180 cm<sup>3</sup> and cross-section 15 cm<sup>2</sup>, its length is 12 cm. Convert 2000 cm<sup>3</sup> to 2 litres, keeping volume conversion separate from surface area.

Use cross-sectional area, not perimeter, in the volume formula. Distinguish cylinder radius from diameter and identify whether end faces are present. A length times an area gives cubic units.

AQA G16/G17 includes cuboids, right prisms and cylinders. Draw the constant end shape, calculate it first and label the extrusion length. Preserve exact multiples of pi when asked.

Constant triangular end area 12 → volume 120



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Circle lengths, areas and composite boundaries · Foundation

Circle circumference is  $2\pi r = \pi d$  and area is  $\pi r^2$ . A semicircle has half the circle area, but its complete perimeter includes the diameter as well as half the circumference. For composite shapes count every exposed boundary once, and separate straight edges from arcs.

$$C = 2\pi r, \quad A = \pi r^2, \quad P_{\text{semi}} = \pi r + 2r$$

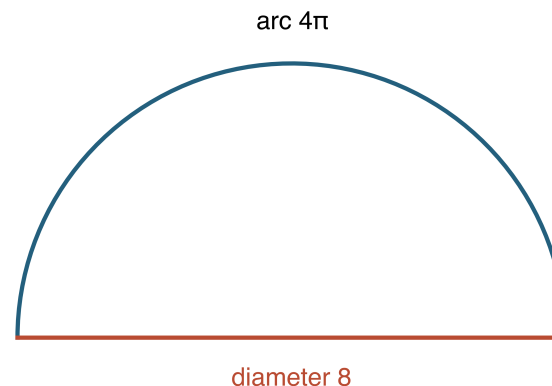
For radius 4 cm, circumference is  $8\pi$  cm and area  $16\pi$  cm<sup>2</sup>. A semicircle of that radius has area  $8\pi$  cm<sup>2</sup> and perimeter  $4\pi + 8$  cm. A rectangular 8 by 3 cm window topped by this semicircle has area  $24 + 8\pi$  cm<sup>2</sup>. Its perimeter is the bottom 8, two vertical sides totalling 6 and the curved top  $4\pi$ :  $14 + 4\pi$  cm. The diameter across the join is internal and is not counted. An annulus with outer radius 5 and inner radius 3 has

area  $\pi(25-9)=16\pi$  cm<sup>2</sup>. Its two circular boundaries together have length  $10\pi + 6\pi = 16\pi$  cm, despite this accidental equality of coefficients; length and area still have different units.

Radius must be halved from a given diameter before squaring. Half a circle's circumference is only its arc, not the complete semicircle perimeter. An internal join is not exposed boundary.

AQA G17 includes exact pi answers and composite circle perimeters/areas. Give both the exact expression and the required final rounded value, keeping the meaning and units clear.

Semicircle perimeter: curved arc + straight diameter



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Spheres, cones, pyramids and frustums · Foundation

Pyramid and cone volumes are one third of base area times perpendicular height. A sphere has volume  $\frac{4\pi r^3}{3}$  and area  $4\pi r^2$ . A cone's curved area is  $\pi r l$  using slant height  $l$ ; its volume uses perpendicular height  $h$ . For a frustum subtract the removed similar solid. Composite surface area counts only exposed faces; joined faces are hidden.

$$V_{\text{cone}} = \frac{1}{3}\pi r^2 h, \quad V_{\text{sphere}} = \frac{4}{3}\pi r^3$$

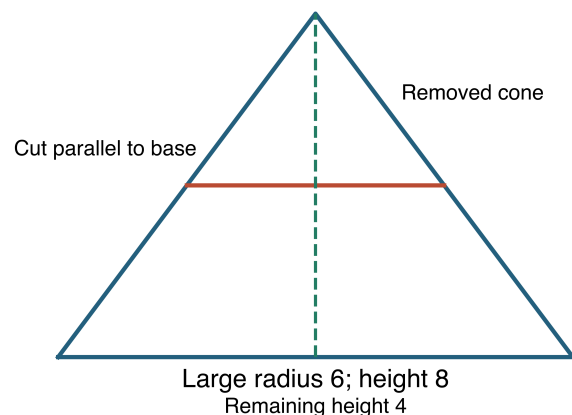
A cone with radius 3 cm and perpendicular height 4 cm has slant height 5 cm. Volume is  $12\pi$  cm<sup>3</sup>, curved area  $15\pi$  cm<sup>2</sup> and total closed area  $24\pi$  cm<sup>2</sup>. A sphere of radius 3 has volume  $36\pi$  cm<sup>3</sup> and area  $36\pi$  cm<sup>2</sup>, with different units. A square pyramid of base side 6 and height 4 has volume  $6^2 \times 4/3 = 48$  cm<sup>3</sup>; each triangular face has slant height  $\sqrt{(4^2 + 3^2)} = 5$ , so lateral area is  $4 \times (6 \times 5/2) = 60$  cm<sup>2</sup> and total area 96 cm<sup>2</sup>. A

large cone  $r=6, h=8$  loses a similar top cone  $r=3, h=4$ : frustum volume is  $96\pi - 12\pi = 84\pi \text{ cm}^3$ . Its slant height is  $10-5=5$ ; curved area is  $60\pi - 15\pi = 45\pi$ , and two circular ends add  $36\pi + 9\pi$  for total  $90\pi \text{ cm}^2$ .

Perpendicular height and slant height are not interchangeable. A frustum is not a full cone of the leftover height. Shared composite faces do not contribute exposed area.

AQA G17 additional Foundation includes spheres, pyramids, cones, composite solids and frustums. Use similarity to find missing removed dimensions, then subtract volumes or exposed areas with matching units.

**Frustum volume = whole cone – removed similar cone**



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Arcs, sectors and reverse angle calculations · Foundation

For an angle  $\theta$  in degrees, fraction of a turn is  $\theta/360$ . Arc length is this fraction of  $2\pi r$  and sector area is this fraction of  $\pi r^2$ . Sector perimeter adds the two radii. Rearrange the same fraction to recover an angle from an arc or an area. Keep degree and length units separate.

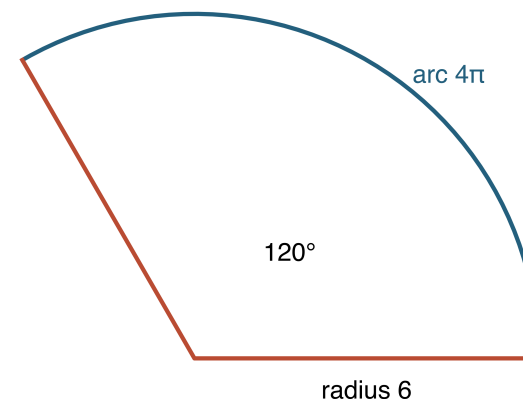
$$s = \frac{\theta}{360^\circ} 2\pi r, \quad A = \frac{\theta}{360^\circ} \pi r^2$$

At  $r=6 \text{ cm}$  and  $\theta=120^\circ$ , the fraction is  $1/3$ : arc is  $4\pi \text{ cm}$ , area is  $12\pi \text{ cm}^2$  and perimeter is  $4\pi + 12 \text{ cm}$ . If another  $r=6$  sector has area  $9\pi \text{ cm}^2$ , its fraction is  $9\pi/36\pi = 1/4$  and angle is  $90^\circ$ . If its arc instead measures  $3\pi \text{ cm}$ , its fraction is  $3\pi/12\pi = 1/4$ , giving the same angle. A full  $360^\circ$  sector has the whole circle area, while the circle boundary has no extra radii. A  $60^\circ$  sector of radius 3 has area  $(1/6) \times 9\pi = 1.5\pi \text{ cm}^2$ .

An arc length is not a sector perimeter. Use the angle as a fraction of  $360^\circ$ , not  $180^\circ$ . Radius is squared only for area, not arc length.

AQA G18 includes arc lengths, sector angles and areas in degree-based geometry. Show the full-circle quantity and fraction before multiplying, then check that the result fits the angle.

**One third of a turn: sector area  $12\pi$**



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Exact trigonometric values from special triangles · Foundation

Bisect an equilateral triangle of side 2 to obtain a 30–60–90 triangle with sides  $1, \sqrt{3}, 2$ . A right isosceles triangle has sides  $1, 1, \sqrt{2}$ . Apply opposite/hypotenuse, adjacent/hypotenuse and opposite/adjacent to derive sine, cosine and tangent. Use degree angles and retain surds exactly;  $\tan 90^\circ$  is undefined.

$$\sin 30^\circ = \frac{1}{2}, \quad \cos 45^\circ = \frac{\sqrt{2}}{2}, \quad \tan 60^\circ = \sqrt{3}$$

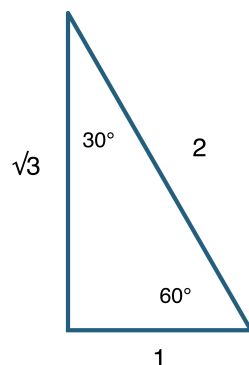
For angles  $0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$ , sine values are  $0, 1/2, \sqrt{2}/2, \sqrt{3}/2, 1$ ; cosine values are  $1, \sqrt{3}/2, \sqrt{2}/2, 1/2, 0$ . For  $0^\circ, 30^\circ, 45^\circ, 60^\circ$ , tangent values are  $0, 1/\sqrt{3}, 1, \sqrt{3}$ . The side 1 opposite  $30^\circ$  in the bisected equilateral triangle gives  $\sin 30^\circ = 1/2$ ; the adjacent  $\sqrt{3}$  gives  $\cos 30^\circ = \sqrt{3}/2$  and  $\tan 30^\circ = 1/\sqrt{3}$ . In the isosceles right triangle,  $\sin 45^\circ = \cos 45^\circ = 1/\sqrt{2} = \sqrt{2}/2$ . A right triangle with hypotenuse 10 and angle  $30^\circ$  has opposite side 5 and adjacent side  $5\sqrt{3}$ . Since  $\sin 90^\circ = 1$  and  $\cos 90^\circ = 0$ , tangent at  $90^\circ$  would divide by zero.

Sine and cosine interchange when the chosen acute angle changes to its complement. Do not turn  $\sqrt{3}$  into a rounded decimal when an exact answer is required. Tangent at

90° is not zero.

AQA G21 is additional Foundation and includes the listed exact values. Derive them from labelled special triangles, then use them in G20 right-triangle calculations.

Bisected equilateral triangle: sides 1,  $\sqrt{3}$ , 2



*Original native-lesson illustration; labels belong to its worked example.*

#### ■ Sine rule, cosine rule and triangle area · Higher

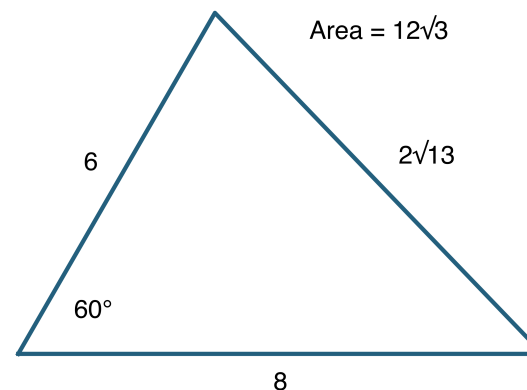
Sine rule pairs opposite sides/angles:  $a/\sin A = b/\sin B = c/\sin C$ . Cosine rule  $a^2 = b^2 + c^2 - 2bc \cos A$  uses the angle opposite  $a$ . Area is  $ab \sin C / 2$  when  $C$  lies between  $a$  and  $b$ . Choose the rule from the known information. An inverse sine may give an acute angle and an obtuse supplement; check the angle sum and supplied sides before accepting either.

$$a^2 = b^2 + c^2 - 2bc \cos A, \quad A_{\Delta} = \frac{1}{2} ab \sin C$$

With sides 6 and 8 enclosing 60°,  $c^2 = 36 + 64 - 96 \times (1/2) = 52$ , hence  $c = 2\sqrt{13}$ . Its area is  $(1/2) \times 6 \times 8 \times \sin 60^\circ = 12\sqrt{3}$ . For  $a=4$  opposite  $A=30^\circ$  and  $B=45^\circ$ ,  $b=4 \sin 45^\circ / \sin 30^\circ = 4\sqrt{2}$ . If sides  $a=7, b=5, c=6$ ,  $\cos A = (25 + 36 - 49) / (2 \times 5 \times 6) = 1/5$ , so  $A \approx 78.5^\circ$ . To find an angle from area 12 with enclosing sides 6 and 8,  $\sin C = 24/48 = 1/2$ ;  $C$  could be 30° or 150° until the remaining data selects a shape. Label opposite pairs and check triangle inequalities to reject impossible side combinations.

Do not pair a side with its adjacent angle in the sine rule. The area angle must be included. A calculator's first inverse-sine answer need not be the only possible triangle. AQA G22/G23 Higher includes unknown sides/angles and areas of general triangles. Write the chosen rule before substitution and state any second possible configuration.

Two sides and their included angle



*Original native-lesson illustration; labels belong to its worked example.*

#### ■ Right triangles in three-dimensional shapes · Higher

Identify a plane containing the wanted length or angle. Calculate a base diagonal first, then combine it with the perpendicular height. For an angle between a line and a plane, use the angle between the line and its orthogonal projection onto that plane. Mark right angles; general-triangle rules are used only when the selected triangle is not right-angled.

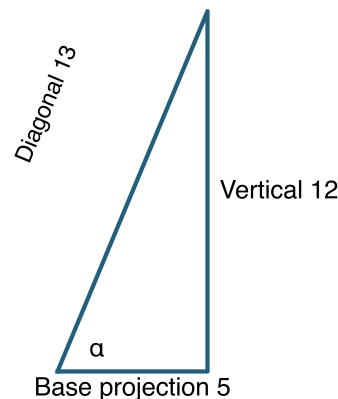
$$d = \sqrt{l^2 + w^2 + h^2}, \quad \tan \alpha = \frac{h}{\sqrt{l^2 + w^2}}$$

A cuboid of width 3, depth 4 and height 12 has base diagonal  $\sqrt{9+16}=5$  and space diagonal  $\sqrt{25+144}=13$ . The angle  $\alpha$  of the space diagonal to the horizontal base satisfies  $\tan \alpha = 12/5$ , giving about 67.4°. Its sine is 12/13; its cosine is 5/13. In a square-based pyramid of side 6 and vertical height 4, the base centre-to-side-midpoint distance is 3, giving face slant height 5. The centre-to-corner distance is  $3\sqrt{2}$ , giving edge length  $\sqrt{(18+16)}=\sqrt{34}$ . The face slant and edge lengths differ because their base projections differ.

Do not combine unrelated lengths as if they met at a right angle. The angle to a plane uses the base projection, not an arbitrary base edge. A pyramid's face slant is different from its sloping edge.

AQA G20 Higher extends right-triangle and, where possible, general-triangle reasoning into 3D. Draw the relevant section triangle separately and name which spatial points it represents.

Cuboid section: first  $\sqrt{3^2+4^2}=5$ , then  $\sqrt{5^2+12^2}=13$



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Vector geometry and midpoint arguments · Higher

Add/subtract components and multiply a vector by a scalar. Position vectors locate points from one origin; displacement  $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$  joins two points. A scalar multiple gives parallel directions; to establish collinearity, also connect the displacements to a common point. Midpoints average position vectors. Give a chain of vector equalities with clear start and end points.

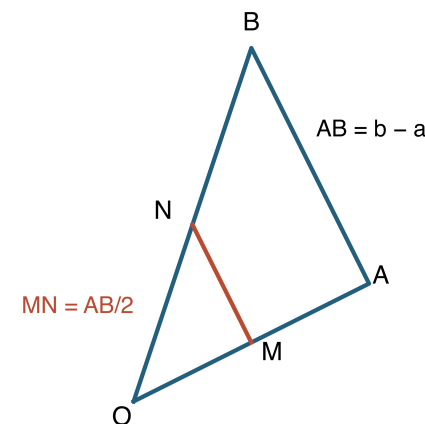
$$\overrightarrow{MN} = \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\overrightarrow{AB}$$

Let  $\overrightarrow{OA} = \mathbf{a}$  and  $\overrightarrow{OB} = \mathbf{b}$ , with M midpoint of OA and N midpoint of OB. Then  $\overrightarrow{OM} = \mathbf{a}/2$  and  $\overrightarrow{ON} = \mathbf{b}/2$ , so  $\overrightarrow{MN} = \mathbf{b}/2 - \mathbf{a}/2 = (\mathbf{b} - \mathbf{a})/2 = \overrightarrow{AB}/2$ . Thus MN is parallel to AB and half as long. For  $\mathbf{a} = (4, 2)$ ,  $\mathbf{b} = (2, 6)$ ,  $\mathbf{M} = (2, 1)$ ,  $\mathbf{N} = (1, 3)$ ,  $\overrightarrow{AB} = (-2, 4)$  and  $\overrightarrow{MN} = (-1, 2)$ , verifying the general result. In parallelogram OACB with  $\overrightarrow{OC} = \mathbf{a} + \mathbf{b}$ , the midpoint of OC is  $(\mathbf{a} + \mathbf{b})/2$ ; the midpoint of AB is the same, so the diagonals bisect each other. If  $\overrightarrow{AP} = 3\overrightarrow{AB}/2$ , P lies on line AB beyond B; if  $\overrightarrow{AP} = \overrightarrow{AB}/2$ , P is its midpoint. A parallel vector at another location alone does not prove three specified points collinear.

Keep  $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ , not  $\mathbf{a} - \mathbf{b}$ . Parallelism alone does not locate a line. A numerical example can check the algebra but cannot replace the general midpoint proof.

AQA G25 Higher uses vectors for geometric arguments/proofs, while G24 and the basic G25 operations also belong to Foundation. This proof lesson avoids scalar products and spatial line equations.

Midpoint segment: half length, parallel direction



*Original native-lesson illustration; labels belong to its worked example.*

## Original independent transfer

### ■ Foundation

A cylindrical container has internal radius 3 cm and height 10 cm. Find its capacity in cubic centimetres and litres. A rectangular face is 6 cm by 8 cm; find its diagonal, naming the condition for your method.

### ■ Foundation worked solution

Cylinder volume  $V = \pi r^2 h = \pi(3 \text{ cm})^2(10 \text{ cm}) = 90\pi \text{ cm}^3$ . Since 1000 cubic centimetres is one litre, capacity is  $0.09\pi \text{ L}$ , about 0.283 L. A rectangular face has perpendicular sides. Pythagoras gives  $d = \sqrt{a^2 + b^2} = \sqrt{(6 \text{ cm})^2 + (8 \text{ cm})^2} = 10 \text{ cm}$ . A diagonal is not an extra edge of the rectangle's perimeter.

### ■ Higher

In a circle, AB is a diameter and C lies elsewhere on the circumference. Angle CAB is 32 degrees. Find angles ACB and ABC. Find the acute angle between the tangent at A and AC, explaining the circle theorem and which side of the tangent you mean.

### ■ Higher worked solution

Angle ACB is 90 degrees because an angle in a semicircle is a right angle. Triangle angles give  $\angle ABC = 180 - 90 - 32 = 58$  degrees. By the alternate segment theorem, the angle between chord AC and the appropriate tangent ray equals angle ABC, 58 degrees. The other angle on the tangent's straight line is 122 degrees. Naming the

acute angle selects 58; the drawing’s apparent scale is unnecessary. The diameter and tangent facts are required hypotheses.

Terms

- scale factor
- hypotenuse
- resultant
- centre of enlargement
- locus
- cyclic quadrilateral
- line of symmetry
- corresponding angles
- congruence
- segment
- midpoint
- elevation
- bearing
- invariant
- angle at the centre
- alternate segment theorem
- perpendicular height
- cross-section
- circumference
- frustum
- sector
- exact value
- included angle
- projection
- collinear

Probability – native teaching supplement

AQA · GCSE · Mathematics

Supported teaching and tier boundary

8300: Probability. Version: Version 1.0, 12 September 2014; first examination 2017. Foundation teaching and Higher additions are labelled below. This reference packages the existing native-lesson crosswalk. It does not certify unreviewed specification rows or a whole qualification. Original diagnostics are separate and are not reproduced.

■ Probability trees and outcomes · Foundation

A probability lies between 0 and 1. Exhaustive, mutually exclusive outcomes have probabilities summing to 1. Multiply successive branch probabilities and add separate routes to an outcome.

$$P(RR) = P(R_1)P(R_2 \mid R_1)$$

A bag contains 3 red and 2 blue counters. With replacement,  $P(\text{two red}) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25} = 0.36$ . Without replacement, the red-red branch is  $\frac{3}{5} \times \frac{2}{4} = 0.3$ . Label each branch before multiplying.

Mutually exclusive means no overlap; independent means that knowing one event does not change the other’s probability. Two disjoint events with positive probability are not independent.

Use a frequency table or a simple tree before calculating. Formal conditional probability formulae are outside this Foundation/Core support lesson.

■ Probability, trees and conditional reasoning · Higher

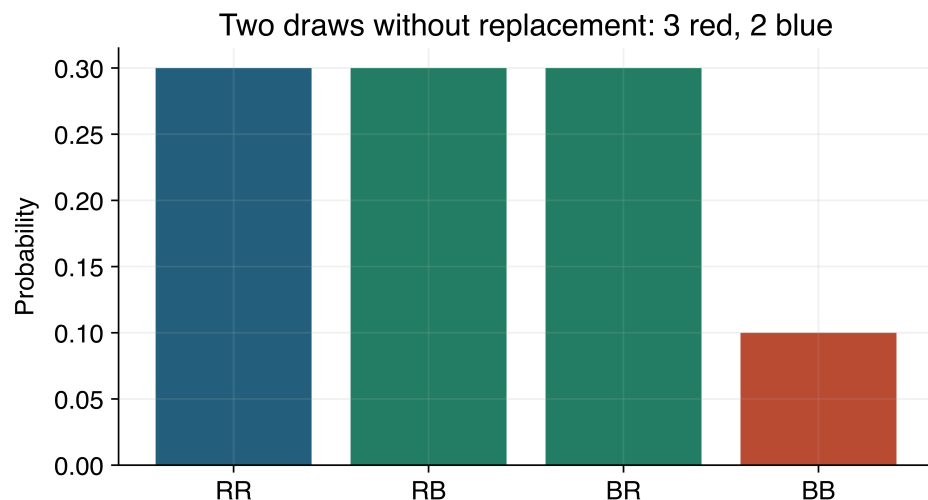
Multiply along a tree branch and add disjoint branches. With replacement, the composition stays fixed. Conditional probability is  $P(A \text{ given } B) = \frac{P(A \cap B)}{P(B)}$ , for  $P(B) > 0$ .

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

Without replacement,  $P(\text{two red}) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$ .  $P(\text{one of each}) = \frac{3}{5} \times \frac{2}{4} + \frac{2}{5} \times \frac{3}{4} = \frac{3}{5}$ . If  $P(A \cap B) = 0.12$  and  $P(B) = 0.3$ ,  $P(A \text{ given } B) = 0.4$ .

Mutually exclusive means no overlap; independent means that knowing one event does not change the other’s probability. Two disjoint events with positive probability are not independent.

A two-way table makes the restricted denominator visible. Before using a product  $P(A)P(B)$ , justify independence from the context or the supplied information.



*Original native-lesson illustration; labels belong to its worked example.*

#### ■ Experimental probability, fairness and expected outcomes · Foundation

Record each trial consistently and total the counts. Relative frequency estimates probability as outcome frequency/trials. For a stated probability  $p$ , expected count in  $n$  future trials is  $np$ ; it is a long-run average, not a guarantee. Unbiased trials with larger samples usually give more stable estimates, but cannot force exact agreement with theory. Probabilities lie from 0 to 1 and a mutually exclusive exhaustive list sums to 1.

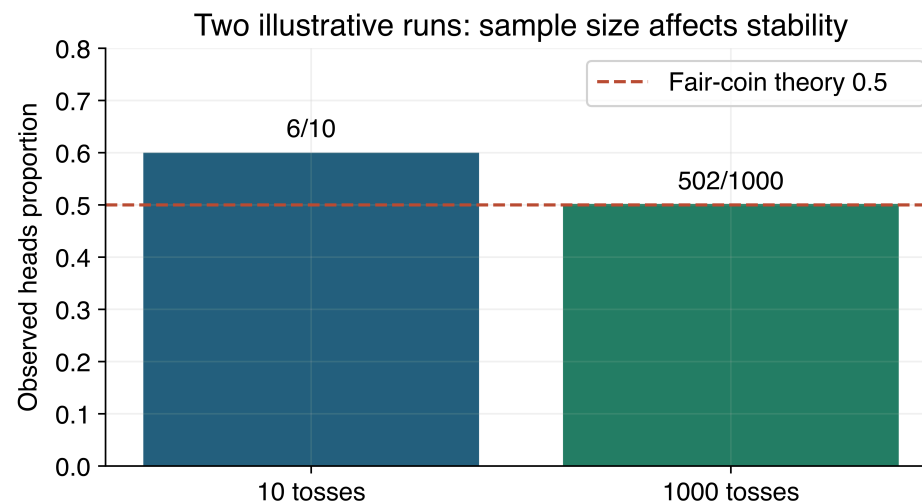
$$\hat{p} = \frac{f}{n}, \quad E = np$$

A spinner lands red 18 times in 60 spins, giving estimated  $P(\text{red}) = 18/60 = 0.3 = 30\%$ . Using this estimate predicts  $0.3 \times 200 = 60$  red results in 200 future spins. If red, blue and green are exhaustive with probabilities 0.3, 0.45 and  $p$ , then  $p = 1 - 0.75 = 0.25$ . A fair coin has theoretical  $P(\text{head}) = 0.5$ ; 100 tosses give expected heads 50, but 48 or 54 is possible. An experiment with 6 heads in 10 tosses estimates 0.6; one with 502 heads in 1000 estimates 0.502. These illustrative runs are not proof that error falls at every stage. Spin using the same method and record all results rather than stopping when a favourite outcome appears.

A larger biased sample can still be misleading. Expected does not mean certain. Mutually exclusive events cannot happen together; if categories overlap, do not simply add their probabilities as separate outcomes.

AQA P1–P5 uses frequency tables/trees, randomness/fairness, expected counts, the probability scale and empirical/theoretical comparison. State whether a value is ob-

served, estimated or theoretical.



*Original native-lesson illustration; labels belong to its worked example.*

#### ■ Systematic possibilities and equally likely outcomes · Foundation

List outcomes systematically using a grid, table or tree. For equally likely outcomes, probability is favourable outcomes/total outcomes. Keep ordered outcomes distinct when the experiments have labelled first and second stages. A grid shows completeness and prevents duplicate counting; unequal probabilities need weights rather than a simple count.

$$P(E) = \frac{\text{favourable equally likely outcomes}}{\text{all equally likely outcomes}}$$

A coin and a fair die have 12 equally likely ordered outcomes: H1 to H6 and T1 to T6. Heads with an even die result has three outcomes, so probability is  $3/12 = 1/4$ . Two dice have 36 ordered pairs. Total seven arises from (1,6), (2,5), (3,4), (4,3), (5,2), (6,1), so probability is  $6/36 = 1/6$ . A double has six outcomes and probability  $1/6$ . Total at least eleven occurs in (5,6), (6,5), (6,6), giving  $1/12$ . With a spinner divided into unequal sectors, the named colours are not automatically equally likely; use the sector proportions or a justified experimental estimate.

Counting totals rather than equally likely dice pairs gives wrong weights. State whether order matters. A possibility table lists outcomes; its entries are not automatically equiprobable.

AQA P6/P7 uses tables, grids and trees to enumerate theoretical possibilities. Explain why the selected elementary outcomes have equal probability before dividing counts.

36 equally likely cells; six sums equal seven

|   |   |   |   |    |    |    |
|---|---|---|---|----|----|----|
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 5 | 6 | 7 | 8 | 9  | 10 | 11 |
| 4 | 5 | 6 | 7 | 8  | 9  | 10 |
| 3 | 4 | 5 | 6 | 7  | 8  | 9  |
| 2 | 3 | 4 | 5 | 6  | 7  | 8  |
| 1 | 2 | 3 | 4 | 5  | 6  | 7  |
|   | 1 | 2 | 3 | 4  | 5  | 6  |

Original native-lesson illustration; labels belong to its worked example.

#### ■ Venn diagrams, unions and two-way counts · Foundation

Place the overlap in a Venn diagram first, then fill the only-regions and neither-region. Union means at least one named set; intersection means both; complement means outside a named set in the stated universal group. Two-way tables classify every observation by one category from each of two variables. Check row, column and grand totals.

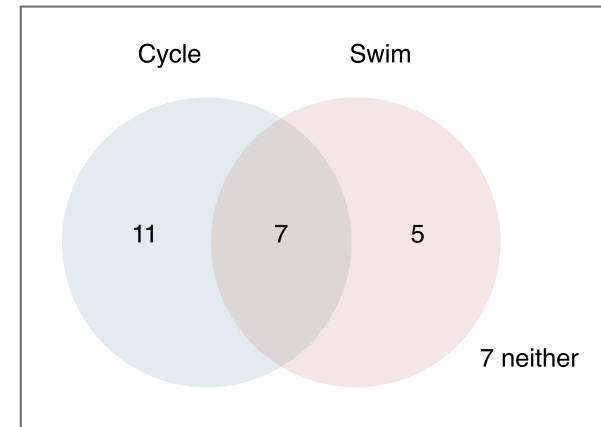
$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

In a class of 30, 18 cycle, 12 swim and 7 do both. Cycling only is 11 and swimming only 5; at least one is  $11+7+5=23$ , leaving 7 neither. A random student has probability  $7/30$  of both and  $23/30$  of at least one. The two-way table has cycle-and-swim 7, cycle-not-swim 11, not-cycle-swim 5 and neither 7. The cycle row totals 18, swim column totals 12 and grand total 30. The outcomes both, cycling only, swimming only and neither are mutually exclusive and exhaustive, so their probabilities sum to 1. A frequency tree starts at 30, branches to cycle 18 and not-cycle 12, then to swim/not-swim counts  $7/11$  and  $5/7$ . Each pair of terminal counts sums back to its parent.

Do not add the overlap twice. Neither is outside both circles, not just outside their overlap. A universal group must be stated before taking a complement.

AQA P4/P6 uses exhaustive events, systematic sets and Venn/table representations. Translate the words both, either/at least one, only and neither into the correct counted regions.

30 students: overlap first, then only and neither



Original native-lesson illustration; labels belong to its worked example.

#### ■ Conditional probability with tables and expected frequencies · Higher

Conditioning restricts the denominator to the given group. In a table, divide the intersection count by the condition's row/column total.  $P(A \text{ given } B) = P(A \text{ and } B) / P(B)$  when  $P(B) > 0$ . Reversing the condition usually changes the denominator. Expected-frequency trees help interpret percentages without treating the two conditions as interchangeable.

$$P(A | B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

Using a class of 30 with 18 cyclists, 12 swimmers and 7 doing both,  $P(\text{swim given cycle}) = 7/18$ , while  $P(\text{cycle given swim}) = 7/12$ . The unconditional swim probability is  $12/30 = 0.4$ . For an expected cohort of 1000, suppose 20% have a condition; a test is positive for 90% with it and 10% without it. Expected positive counts are 180 from 200 with the condition and 80 from 800 without it. Of 260 positive tests, the conditional proportion with the condition is  $180/260 = 9/13 \approx 0.6923$ , not 90%. These are stated illustrative model rates, not claims about a real diagnostic test.

Given positive and positive given condition are different questions. Use the conditioned group total, not the grand total. A rare starting category can make false-positive counts significant even when the detection rate is high.

AQA P9 Higher requires conditional calculations and interpretation using two-way tables, trees and Venn diagrams. Name the restricted group and retain expected counts until the final ratio.

Given positive: use 180 out of 260, not 180 out of 200

|              | Positive | Negative | Total |
|--------------|----------|----------|-------|
| Condition    | 180      | 20       | 200   |
| No condition | 80       | 720      | 800   |
| Total        | 260      | 740      | 1000  |

*Original native-lesson illustration; labels belong to its worked example.*

## Original independent transfer

### ■ Foundation

A bag has three red and two blue counters. Two are drawn without replacement. Make a complete possibility tree and find the probability of different colours. Explain how the second-stage denominators change.

### ■ Foundation worked solution

First probabilities are R:3/5, B:2/5. After R, the second probabilities are R:2/4, B:2/4. After B they are R:3/4, B:1/4. Terminal probabilities RR, RB, BR, BB are 6/20, 6/20, 6/20, 2/20 and sum to one. Different colours are the disjoint RB and BR paths, so probability is  $6/20 + 6/20 = 3/5$ . Multiplying stages follows a path; adding combines mutually exclusive paths. Replacement would change the model.

### ■ Higher

In 50 students, 30 study art, 25 study music and 15 study both. Find music given art and art given music. Are the events independent? Reconcile the counts using a two-way table.

### ■ Higher worked solution

Both=15, art only=15, music only=10 and neither=10; their sum is 50.  $P(M | A) = 15/30 = 1/2$  while  $P(A | M) = 15/25 = 3/5$ . The intersection probability is  $15/50 = 3/10$ , equal to  $(30/50)(25/50) = 3/10$ , so these events are independent in this finite model. Their unequal conditional probabilities do not contradict independence, since independence compares each condition with the matching unconditional probability:  $P(M) = 1/2$  and  $P(A) = 3/5$ .

## Terms

conditional probability .

relative frequency .

sample space .

intersection .

# Statistics – native teaching supplement

AQA · GCSE · Mathematics

## Supported teaching and tier boundary

8300: Statistics. Version: Version 1.0, 12 September 2014; first examination 2017.

Foundation teaching and Higher additions are labelled below. This reference packages the existing native-lesson crosswalk. It does not certify unreviewed specification rows or a whole qualification. Original diagnostics are separate and are not reproduced.

### ■ Centre, spread and data displays · Foundation

The mean is total divided by count. The median is the central value after sorting. The range is maximum minus minimum. Use frequency tables, bar charts and suitable comparisons.

$$\bar{x} = \frac{\sum x_i}{n}$$

For 2,4,4,6,9, total=25 and count=5, so mean=5. The central value is 4, so median=4. The range is 9-2=7. Explain both a typical value and the spread.

The tallest histogram bar need not contain the most observations. A grouped mean is an estimate. Correlation does not prove causation, and extrapolation extends beyond the observed range.

A bar chart uses separate bars for categories. Unequal-class-width histograms and formal density calculations are outside this Foundation/Core lesson.

### ■ Data summaries, histograms and interpretation · Higher

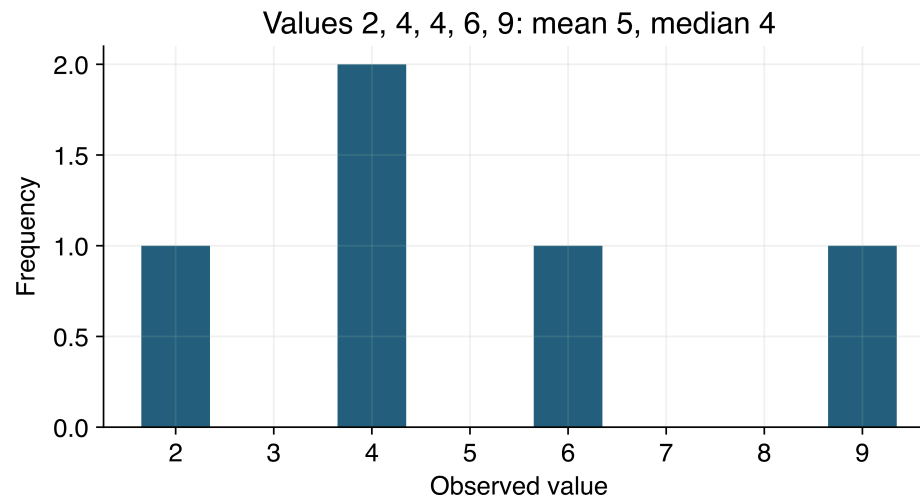
Compare an appropriate average and spread in context. A histogram uses area for frequency, so height=frequency/class width. Grouped estimates assume representative values within intervals.

$$\bar{x} = \frac{\sum x_i}{n}, \quad \text{density} = \frac{\text{frequency}}{\text{class width}}$$

A class from 10 to 20 with frequency 30 has density 30/10=3. A class from 20 to 40 with frequency 20 has density 20/20=1. Its wider bar must not be mistaken for a larger density. For values 2,4,4,6,9, the median is 4 and mean is 5.

The tallest histogram bar need not contain the most observations. A grouped mean is an estimate. Correlation does not prove causation, and extrapolation extends beyond the observed range.

Choose a display that fits the data type. Give both a numerical comparison and what it means for the population; do not infer more precision than the sample supports.



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Cumulative frequency and box plots · Higher

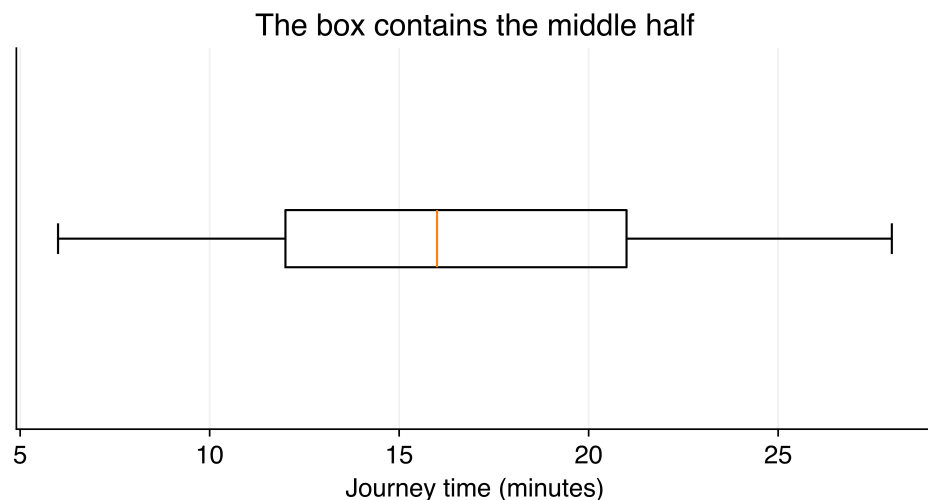
A cumulative frequency counts observations below successive class boundaries. Read quartiles at one quarter, one half and three quarters of the total frequency. A box plot represents minimum, lower quartile, median, upper quartile and maximum.

$$\text{IQR} = Q_3 - Q_1$$

For 80 observations, read Q1 at cumulative frequency 20, median at 40 and Q3 at 60. If Q1=12, Q3=21, then IQR=9. Compare the medians for typical journey time and the IQRs for consistency.

Plot against class boundaries rather than midpoints. Grouped quartiles are estimates. The range is sensitive to extremes; the IQR describes only the middle half.

Explain a comparison in the context of the measured quantity. An outlier rule may use Q1-1.5IQR and Q3+1.5IQR; use the rule specified in the task rather than assuming every graph follows it.



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Samples, populations and justified comparisons · Foundation

Define the target population and variables before sampling. A sample is a subset; a census includes the whole population. Random selection reduces systematic selection bias but does not eliminate sampling variability or non-response. Primary data is collected for the present investigation; secondary data was collected by another source or purpose. Discrete data is counted; continuous data is measured.

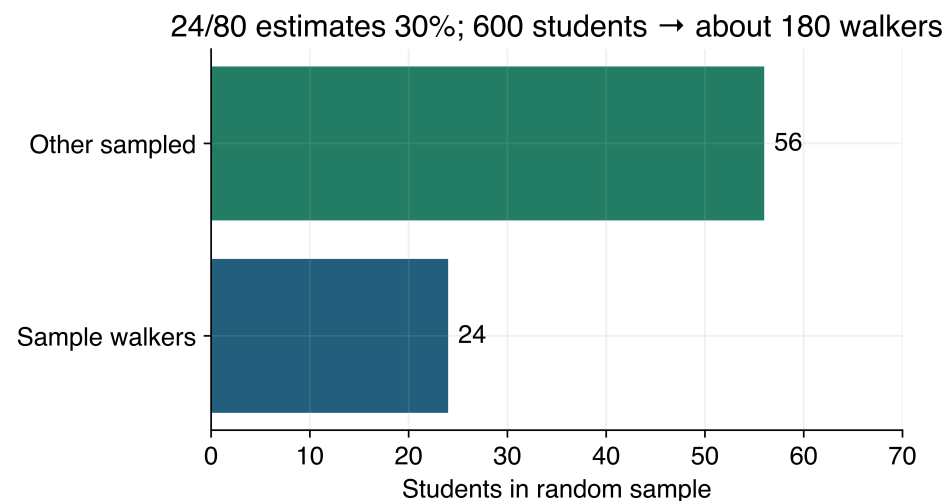
$$\hat{N} = N \frac{f}{n}$$

In a random sample of 80 students, 24 walk to school, so the observed proportion is  $24/80=0.3$ . Applied to a school of 600 it suggests about 180 walkers, with sampling uncertainty: it is an estimate rather than an exact count. A sports-club convenience sample may overrepresent active students. A voluntary online poll can miss people who do not respond; adding responses does not necessarily remove that bias. A study should record who could be selected, missing responses and the question wording. Number of siblings is discrete; travel time is continuous even if recorded to whole minutes. A school's published attendance records are secondary data for a new project; measuring new travel times produces primary data.

Sample size alone cannot fix biased selection. A precise calculated estimate need not be accurate for the target population. Recording continuous measurements as integers does not change the underlying variable type.

AQA S1/S4/S5 uses samples to describe populations and recognises limitations/data types. State what the sample supports and what might make extrapolating to the

whole population unreliable.



*Original native-lesson illustration; labels belong to its worked example.*

### ■ Choosing displays, pie charts and time series · Foundation

Use separate bars for categories, vertical line charts for discrete numerical values and time-ordered lines for a time series. Pie-chart sector angle is  $\text{frequency}/\text{total} \times 360^\circ$ . A pictogram needs a stated key and honest fractional symbols. Label axes/units and choose a scale that does not conceal the relevant variation. Continuous grouped distributions need histograms rather than categorical bars.

$$\theta = \frac{f}{n} 360^\circ$$

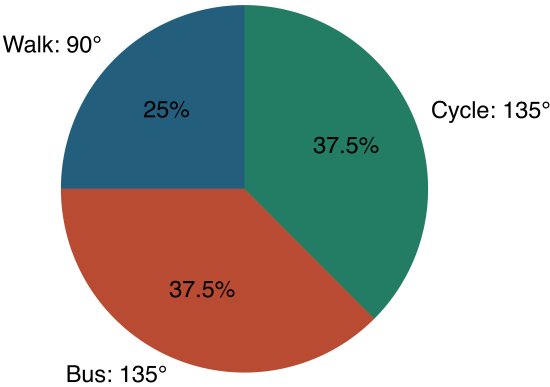
Of 40 students, 10 walk, 15 take the bus and 15 cycle. Pie angles are  $90^\circ, 135^\circ, 135^\circ$ , summing to  $360^\circ$ . With one pictogram symbol representing five students, the groups use 2,3,3 symbols. If one symbol instead means four students, ten walkers need 2.5 symbols. A daily attendance time series 32,35,33,36,34 can be joined in weekday order; the largest count is 36 and its range is 4. A frequency table for number of siblings 0,1,2,3 uses those numbers as positions on a vertical line chart, rather than treating widths as probabilities.

A pie chart must represent a complete total with non-overlapping categories. Pictogram keys control the count, not decorative icon size. A truncated axis may exaggerate a small difference; read the scale before comparing.

AQA S2 covers tables, categorical bars/pies/pictograms, discrete vertical lines and time-series tables/graphs. Explain why the representation fits the variable and the

question.

40 students: frequencies determine sector angles



Original native-lesson illustration; labels belong to its worked example.

Frequency summaries and grouped estimates · Foundation

For exact value frequencies, mean is  $\text{sum}(\text{value} \times \text{frequency}) / \text{total frequency}$ . Locate the median using cumulative counts. For interval data, use class midpoints to estimate a mean; the exact values are unknown. The modal class has greatest frequency, which need not be the tallest unequal-width histogram bar. Compare a suitable average and spread, in context, and state the limitations of grouping or outliers.

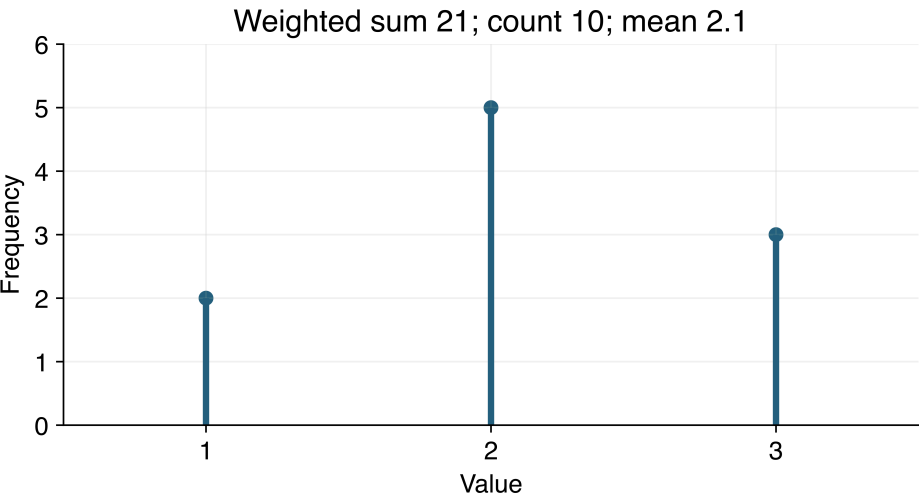
$$\bar{x} \approx \frac{\sum fm}{\sum f}$$

Values 1,2,3 with frequencies 2,5,3 give total 10 and weighted sum  $2+10+9=21$ , so mean is 2.1. The fifth and sixth observations are both 2, giving median 2 and mode 2. For continuous classes  $0 \leq x < 10, 10 \leq x < 20, 20 \leq x < 30$  with frequencies 2,5,3, midpoints 5,15,25 give estimated sum  $10+75+75=160$  and estimated mean 16. The modal class is  $10 \leq x < 20$ . The range of individual observations cannot be recovered exactly from these intervals. For values 2,4,4,6,24, mean is 8 and median 4; the unusually large value raises the mean. Comparing two groups should describe both a typical value and variation, rather than selecting whichever summary favours a claim.

The grouped mean is an estimate because all members are represented by a midpoint. A median is not found by averaging the class labels. Outliers can alter the mean/range substantially.

AQA S4/S5 Foundation includes appropriate mean/median/mode/modal class and

range, with grouped data and outlier awareness. Higher quartiles/box plots are taught separately.



Original native-lesson illustration; labels belong to its worked example.

Scatter graphs, correlation and cautious predictions · Foundation

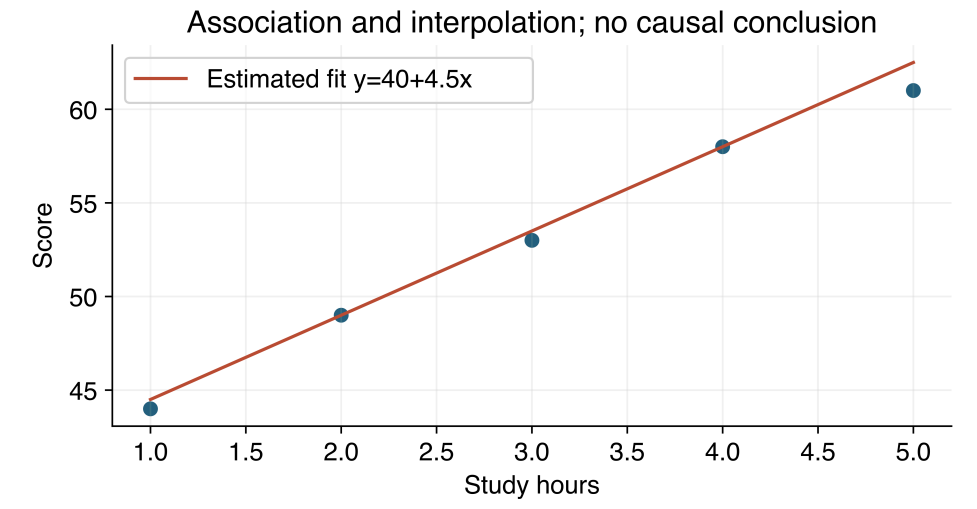
Plot paired observations as points, with one variable on each axis. Positive correlation rises, negative falls and no correlation has no clear trend. Strong/weak describes how tightly points follow a trend, not its steepness. Draw an estimated line of best fit through the middle of the pattern, balancing points around it; it need not pass through the origin. Interpolation stays within observed inputs; extrapolation goes outside them.

$$\hat{y} = 40 + 4.5x$$

Observed study hours 1,2,3,4,5 with scores 44,49,53,58,61 show a positive trend. An estimated line  $y=40+4.5x$  predicts 53.5 at  $x=3$  and 56.2 at  $x=3.6$ . These inputs are inside 1–5, so the predictions interpolate. At  $x=10$  the line gives 85, but this extrapolation may fail if gains flatten or the group differs. Sleep hours and tiredness may show a negative correlation; an almost horizontal cloud with no ordered pattern may show little correlation. A shared cause such as prior preparation can affect both study time and score; the plot alone does not establish causation.

Do not join scatter points in observation order as though they formed a time series. A steep line need not imply strong correlation. An extrapolated formula value is a prediction whose assumptions need scrutiny.

AQA S6 Foundation includes correlation, estimated best-fit lines, predictions and dangers of extrapolation. State the observed input range, association direction/strength and the limits of a causal claim.



Original native-lesson illustration; labels belong to its worked example.

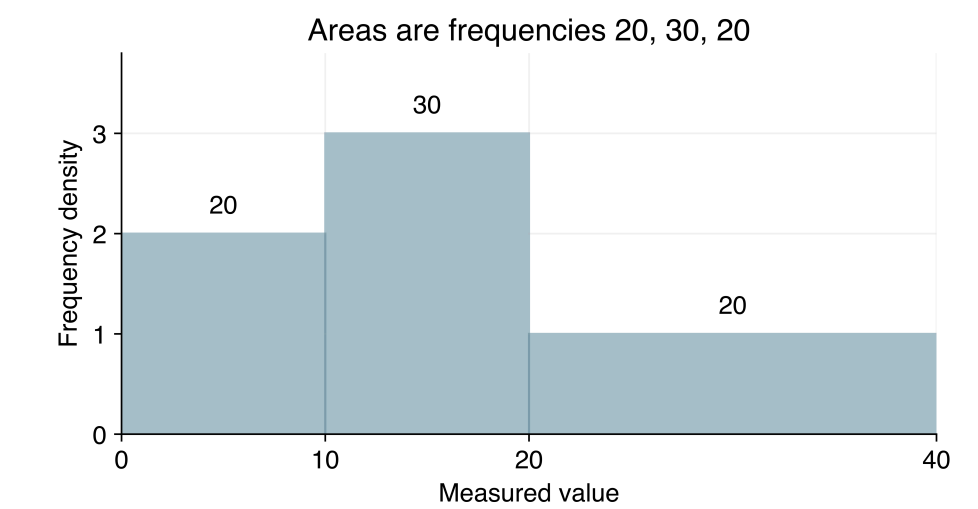
■ **Histogram density and cumulative-frequency construction · Higher**  
 Use continuous adjoining class boundaries on the horizontal axis. Density=frequency/class width, so bar area recovers frequency. Unequal widths require this adjustment. A cumulative-frequency graph plots each upper class boundary against the running total, beginning at the first lower boundary with zero. Read percentiles at fractions of the total; values within classes are estimates.

$$d = \frac{f}{w}, \quad f = dw$$

For classes  $0 \leq x < 10, 10 \leq x < 20, 20 \leq x < 40$  with frequencies 20,30,20, widths are 10,10,20 and densities 2,3,1. The last two bars have different heights but areas 30 and 20. Cumulative points are (0,0),(10,20),(20,50),(40,70). The median is at cumulative count 35; a straight-line estimate in the second class gives  $10 + (35 - 20)/30 \times 10 = 15$ . Q1 is at 17.5 and Q3 at 52.5, giving corresponding interpolated estimates 8.75 and 22.5. Class midpoints 5,15,30 give estimated mean  $(100 + 450 + 600)/70 \approx 16.43$ . A sketch that joins upper boundaries with a smooth curve can give slightly different readings, so use the stated precision and graph.

A histogram uses density on its vertical axis when widths differ. Plot cumulative totals at boundaries, not midpoints. Grouped percentiles depend on interpolation assumptions and are not exact individual measurements.

AQA S3 Higher includes equal/unequal-class histograms and cumulative-frequency graphs, with suitable interpretation. Explain why bar area represents frequency and label the graph axes/units.



Original native-lesson illustration; labels belong to its worked example.

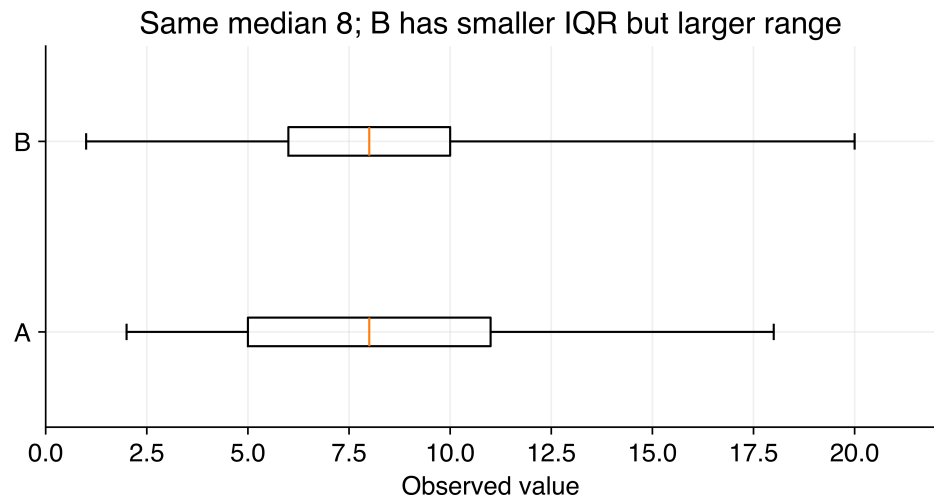
■ **Quartiles, box plots and distribution comparisons · Higher**  
 A box plot marks minimum, lower quartile Q1, median, upper quartile Q3 and maximum, or identifies separately shown outliers according to the stated convention. IQR=Q3-Q1 describes the middle half. Compare median and IQR in context; a smaller IQR suggests less middle-half variation, not necessarily a smaller full range. Quartile conventions vary for a finite raw list, so follow the task or supplied summaries.

$$IQR = Q_3 - Q_1$$

Group A has summary 2,5,8,11,18 and group B has 1,6,8,10,20. Both medians are 8. A has IQR=6 and range=16; B has IQR=4 and range=19. B's middle half is more consistent although its full range is larger. Under a stated  $1.5 \times IQR$  outlier rule, A's fences are  $5 - 9 = -4$  and  $11 + 9 = 20$ ; a new value 23 lies above the upper fence. This rule is a specified convention, not a compulsory rule for every box plot. When using cumulative frequency, read Q1 at  $N/4$  and Q3 at  $3N/4$ , then subtract; small reading errors affect the estimated IQR.

The median line need not be halfway along the box. Equal medians do not mean identical distributions. Whisker meanings depend on whether outliers are drawn separately; read the stated convention.

AQA S4 Higher adds box plots, quartiles and IQR to distribution comparisons. Name both centre and spread, with units and context, and avoid claims about all observations from just the box.



Original native-lesson illustration; labels belong to its worked example.

Original independent transfer

■ Foundation

A sample contains ten values with mean 6 and fifteen values with mean 10. Find the combined mean. A voluntary online survey suggests 70% of respondents cycle; explain why a much larger voluntary response need not establish that 70% of the whole school cycles.

■ Foundation worked solution

The totals are 60 and 150, so combined mean is  $\bar{x} = (10 \cdot 6 + 15 \cdot 10) / (10 + 15) = 210 / 25 = 8.4$ . Weight each group mean by its size. A voluntary survey can overrepresent people interested in cycling and miss nonrespondents. More responses reduce some random variation but do not automatically remove selection bias. State the target population and how students could be selected before generalising.

■ Higher

Grouped continuous data have classes  $0 \leq t < 10$ ,  $10 \leq t < 30$ ,  $30 \leq t < 35$ , with frequencies 8,12,10. Give histogram heights, cumulative frequencies and an estimated mean. Explain why the estimates cannot recover every original observation.

■ Higher worked solution

Class widths are 10,20,5. Frequency density is frequency divided by width, giving 0.8,0.6,2; bar areas return 8,12,10. Cumulative frequencies at upper class boundaries 10,30,35 are 8,20,30, starting from (0,0). Midpoints 5,20,32.5 give estimated total  $8(5) + 12(20) + 10(32.5) = 605$ , hence estimated mean  $605/30 = 121/6$ , about 20.17. Within-class positions are unknown; the midpoint model estimates their total. Height alone is not frequency when widths differ, and a cumulative curve depends on an interpolation assumption between boundaries.

Terms

- range .
- frequency density .
- interquartile range .
- population .
- time series .
- modal class .
- correlation .