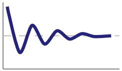
Probability and Random Variables

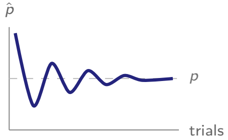
AP Statistics ▪ Unit 4

AP Statistics



What we will cover

1. Randomness and probability
2. Probability rules
3. Conditional and independence
4. Random variables
5. Combining variables
6. Binomial and geometric



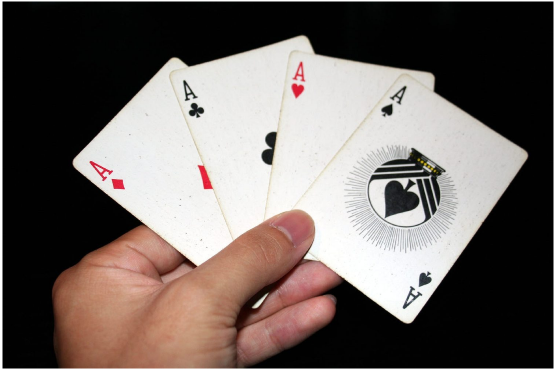
1 Randomness

Randomness and probability

Probability 概率 is the long-run **relative frequency** 相对频率 of an outcome.

- short runs vary by **chance** 偶然
- **simulation** 模拟 estimates a hard probability

Law of large numbers 大数定律: more trials $\Rightarrow \hat{p}$ settles toward the true p.



Playing cards

2 Rules

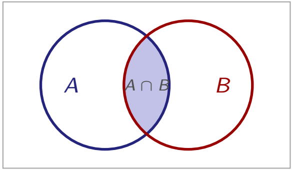
Probability rules

From the **sample space** 样本空间, every **event** 事件 obeys $0 \leq P \leq 1$, summing to 1.

$$P(A^c) = 1 - P(A)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

> **Mutually exclusive** 互斥 events share nothing, so $P(A \cap B) = 0$ and the last term drops.



3 Conditional

Conditional probability and independence

A **conditional probability** 条件概率 restricts to the times B happens:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

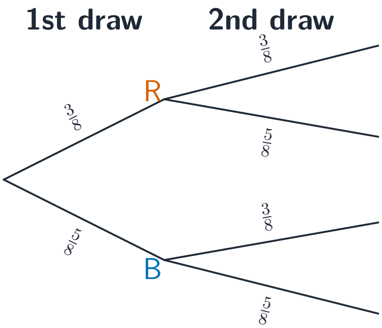
The **multiplication rule** 乘法法则: $P(A \cap B) = P(A)\,P(B \mid A)$. The chance of A once you know B happened. Two-way tables make these easy: restrict to B's row or column, then find A's share.

> **Independent** 独立: $P(A \mid B) = P(A)$, so $P(A \cap B) = P(A)P(B)$. Not the same as mutually exclusive!

	girl	boy
sport	24	36
none	16	24

$P(\text{girl} \mid \text{sport}) = \frac{24}{60}$

On a tree diagram



$$(\mathrm{R},\ \mathrm{R}) = \frac{3}{8} \times \frac{3}{8} = \frac{9}{64}$$

$$(\mathrm{R},\ \mathrm{B}) = \frac{15}{64}$$

$$(\mathrm{B},\ \mathrm{R}) = \frac{15}{64}$$

$$(\mathrm{B},\ \mathrm{B}) = \frac{25}{64}$$

On a tree diagram, multiply the probabilities along the branches

4 Random variables

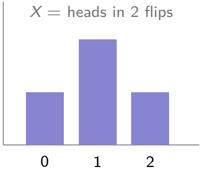
Random variables

A **random variable** 随机变量 X takes a number from a chance process (**discrete** 离散 or **continuous** 连续).

- its **distribution** 概率分布 lists each x with $P(x)$, summing to 1

$$\mu_X = \sum x_i P(x_i) \qquad \sigma_X^2 = \sum (x_i - \mu_X)^2 P(x_i)$$

> μ_X is the **expected value** 期望值 – the long-run average outcome.



Randomness, conditional probability, and expected value

random 随机	individual outcomes are uncertain, but a regular pattern emerges over many repetitions
conditional probability 条件概率	$P(A \mid B) = \dfrac{P(A \text{ and } B)}{P(B)}$
what it means	the chance of A among the cases where B happened – B becomes the new denominator
independence	A and B are independent when $P(A \mid B) = P(A)$
a probability distribution 概率分布	lists every value of a random variable with its probability; they sum to 1
the mean (expected value) 期望值	$\mu_X = \sum x_i P(x_i)$ – the long-run average, not a value you expect to see

An expected value need not be attainable: the expected number of heads in three tosses is 1.5. It is an average over many repetitions.

Estimating Probabilities Using Simulation

A **simulation 模拟** imitates a chance process using random digits or technology.

> Steps: state the model, assign digits to outcomes, run many trials, and record the proportion of trials meeting the condition.

5 Combining

Combining random variables

A **linear transformation** 线性变换:

$$\mu_{aX+b} = a\mu_X + b \qquad \sigma_{aX+b} = |a|\,\sigma_X$$

For a **sum or difference** 和或差 of **independent** 独立 variables, means combine and variances add – for *both*:

$$\mu_{X\pm Y} = \mu_X \pm \mu_Y \qquad \sigma^2_{X\pm Y} = \sigma^2_X + \sigma^2_Y$$

> Never subtract variances for a difference – variances always add.

Two packages

$\sigma_X = 0.3$, $\sigma_Y = 0.4$, independent.

$$\sigma_{X+Y} = \sqrt{0.3^2 + 0.4^2} = 0.5$$

The difference $X - Y$ has the *same* $\sigma = 0.5$.

6 Binomial

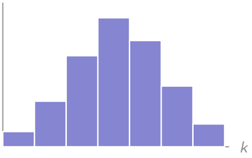
The binomial distribution

The **binomial setting** 二项分布情境 counts successes – remember **BINS**: **B**inary, **I**ndependent, fixed **N**umber, **S**ame p.

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

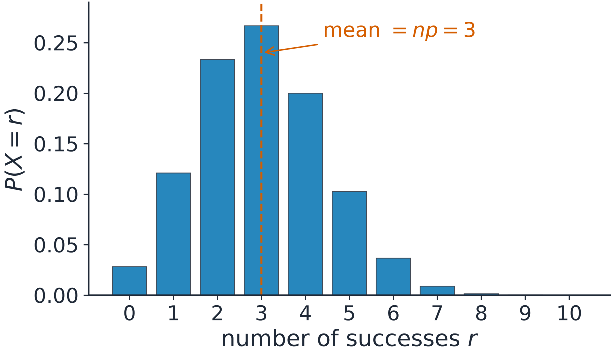
$$\mu_X = np \qquad \sigma_X = \sqrt{np(1 - p)}$$

> Shape: symmetric at $p = 0.5$, skewed otherwise; more bell-shaped as n grows.



Binomial: count of successes.

Introduction to the Binomial Distribution



The binomial distribution, with mean n times p

Worked example – binomial by hand

A free-throw shooter makes 80%. In $n = 10$ attempts, find $P(X = 8)$ and describe the distribution's centre and spread.

$$P(X = 8) = \binom{10}{8}(0.8)^8(0.2)^2$$

$$= 45(0.1678)(0.04) \approx 0.302$$

$\mu = np = 10(0.8) = 8$ makes

$\sigma = \sqrt{10(0.8)(0.2)} \approx 1.26$

$\binom{10}{8} = 45$ counts which 8 shots dropped – forget it and you find one single ordering (0.0067), not the event. Check **BINS** first: without independence it is not binomial at all.

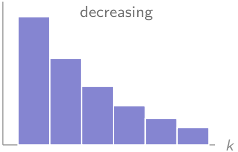
7 Geometric

The geometric distribution

The **geometric setting** 几何分布情境 keeps going *until* the first success (no fixed n).

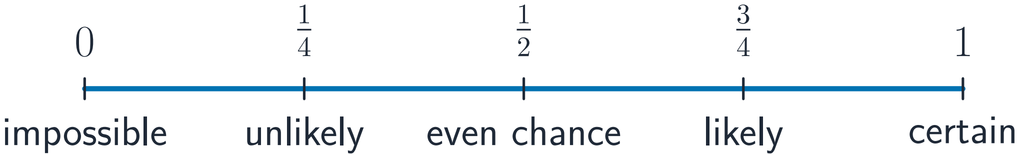
$$P(X = k) = (1 - p)^{k-1}\, p \qquad \mu_X = \frac{1}{p}$$

> X counts trials until the first success; a binomial counts successes in a fixed number of trials.



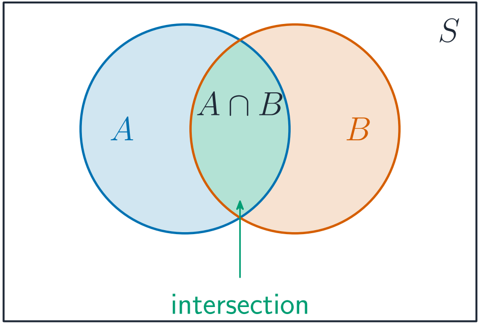
Roll until a six: $\mu = 6$.

Introduction to Probability



> Probability runs from 0 (impossible) to 1 (certain)

Mutually Exclusive Events



A Venn diagram: the overlap is the intersection of two events

Independent Events and Unions of Events

die 2

die 1

+	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

the six highlighted cells all give a total of 7

> A sample space diagram lists every equally likely outcome

Exam tips

- A probability lies in $[0, 1]$; use the **complement** $(1 - P)$ and add mutually exclusive events.
- For independent events multiply; for "and/or" use the general addition and conditional rules.
- **Expected value** $= \sum(\text{value} \times \text{probability})$.
- Recognise **binomial** (fixed n, two outcomes, constant p) and **geometric** settings.
- Draw a tree or table for multi-stage problems and multiply along branches.

8 Recap

Unit 4 – key takeaways

1 Probability

long-run relative frequency; law of large numbers

2 Rules

addition for or; multiplication for and; conditional

3 Random vars

$\mu = \sum xP(x)$; variances add for independent

4 Distributions

binomial $\mu = np$; geometric $\mu = 1/p$

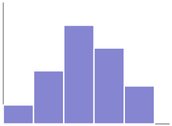
Check yourself

1. Write the complement rule for $P(A)$.

2. For independent X, Y, how do you combine their variances?

3. What does BINS stand for?

4. A binomial has $n = 20,\ p = 0.5$: what is μ_X?



Answers **1.** $P(A^c) = 1 - P(A)$ **2.** add them: $\sigma_X^2 + \sigma_Y^2$ **3.** Binary, Independent, Number, Same p **4.** $np = 10$