

Inference for Categorical Data: Chi-Square

AP Statistics

Are My Results Unexpected?

When data are **counts** spread across several categories, we test whether the observed counts differ from what a claim predicts. The tool is the **chi-square** 卡方 (χ^2) statistic, which adds up the standardized gaps between observed and expected counts:

$$\chi^2 = \sum \frac{(\text{observed} - \text{expected})^2}{\text{expected}}.$$

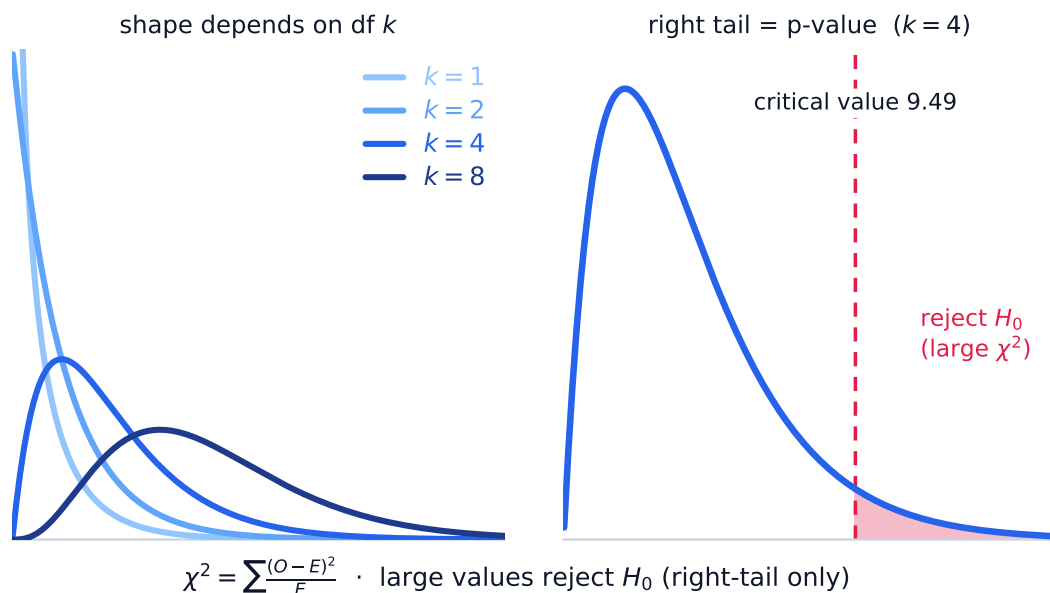
A large χ^2 means the observed counts are far from expected – evidence against the claim. The **chi-square distribution** is right-skewed and depends on its **degrees of freedom** 自由度.

Setting Up a Goodness-of-Fit Test

A **goodness-of-fit (GOF)** 拟合优度 test checks whether one categorical variable follows a **claimed distribution** (e.g. “the die is fair”). Hypotheses:

H_0 : the distribution is as claimed H_a : at least one proportion differs.

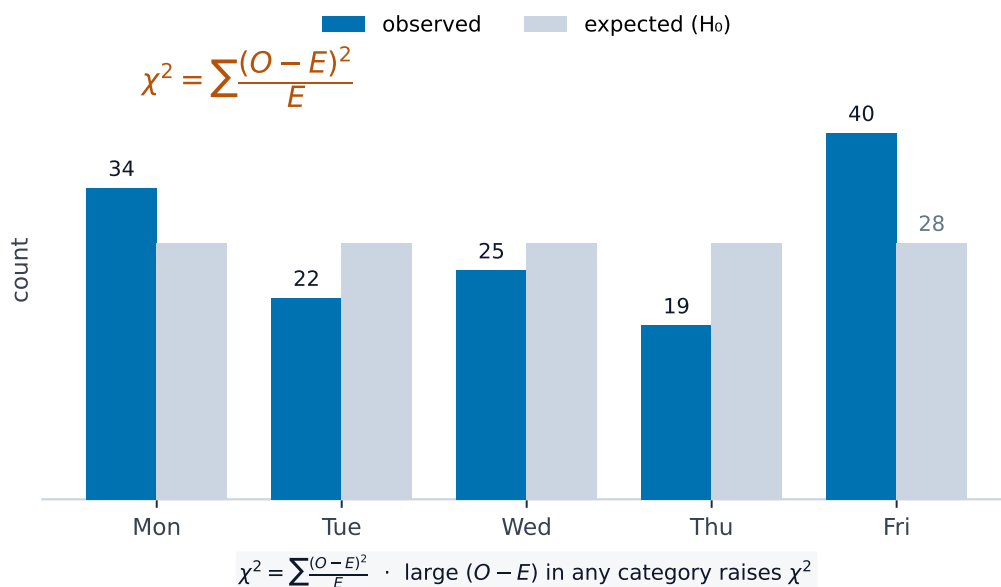
Expected count for each category = $n \times$ (claimed proportion). Conditions: random sample, all **expected counts** ≥ 5 , and the 10% condition.



The chi-square distribution is right-skewed. A large statistic lands in the shaded right tail past the critical value – that is where you reject the model.

Carrying Out a Goodness-of-Fit Test

Compute $\chi^2 = \sum \frac{(O - E)^2}{E}$ with $df = (\text{number of categories}) - 1$. Find the p -value from the chi-square distribution (upper tail), compare to α , and conclude in context. A large component of the sum points to the category that deviates most.



Chi-square compares observed counts with those expected under the null hypothesis

Worked example. A die rolled 60 times gives counts 8, 10, 12, 9, 11, 10. If it is fair, each expected count is $60/6 = 10$, so

$$\chi^2 = \frac{(8 - 10)^2}{10} + \frac{(10 - 10)^2}{10} + \frac{(12 - 10)^2}{10} + \frac{(9 - 10)^2}{10} + \frac{(11 - 10)^2}{10} + \frac{(10 - 10)^2}{10} = 0.4 + 0 + 0.4 + 0.1 + 0.1 + 0 = 1.0,$$

with $df = 6 - 1 = 5$. Write out **every** category, including the two that match their expected count exactly and so add 0 – the sum runs over all six categories, and df counts categories, not just the ones that differ. This χ^2 is small (a large p -value), so we **fail to reject** H_0 – no evidence the die is unfair.

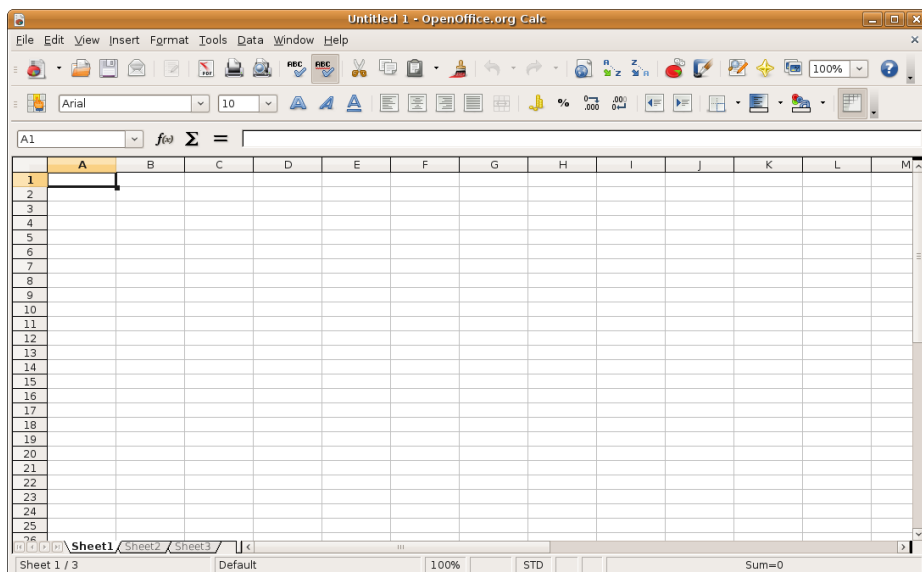
Expected Counts in Two-Way Tables

For a two-way table, the **expected count** in a cell (under “no association”) is

$$E = \frac{(\text{row total}) \times (\text{column total})}{\text{grand total}}.$$

This is the count you would see if the row and column variables were unrelated.

Worked example. In a two-way table a cell's row total is 40, its column total is 50, and the grand total is 200. Its expected count is $E = \frac{40 \times 50}{200} = 10$. Repeating for every cell gives the expected table to compare against the observed one.



A spreadsheet organises categorical counts before a chi-square test

Homogeneity or Independence?

Two tests use the same χ^2 math but answer different questions:

- **Test for homogeneity** 同质性: are the distributions of one categorical variable the **same across several populations or groups** (separate samples/treatments)?
- **Test for independence** 独立性: are two categorical variables **associated within a single population** (one sample, two variables measured)?

The design (several samples vs one sample) decides which name and hypotheses to use.

Carrying Out a Test for Homogeneity or Independence

Compute expected counts, then $\chi^2 = \sum \frac{(O - E)^2}{E}$ over all cells, with

$$df = (\text{rows} - 1)(\text{columns} - 1).$$

Conditions: random data, all expected counts ≥ 5 , 10% condition. Find the p -value, compare to α , and conclude in context – evidence of a difference between groups (homogeneity) or of an association (independence).

Choosing the Right Categorical Procedure

Decide by the setup: one categorical variable against a claimed distribution $\Rightarrow$ **goodness-of-fit**; one sample cross-classified by two variables $\Rightarrow$ **independence**; several samples/groups compared $\Rightarrow$ **homogeneity**. Comparing just **two** proportions can use either a two-proportion z -test or a chi-square test, but only for a **two-tailed** alternative, where they agree exactly ($\chi^2 = z^2$). A chi-square test is always two-tailed, so it cannot give a directional conclusion: if H_a is one-tailed (say $p_1 > p_2$), use the z -test.

Exam tips

- Use $\chi^2 = \sum \frac{(O-E)^2}{E}$ for **categorical** data; always divide by the **expected** count.
- Pick the right test: goodness-of-fit (one variable), independence, or homogeneity (two-way table).
- Compute expected counts as $\frac{\text{row total} \times \text{column total}}{\text{grand total}}$ and check each is ≥ 5 .
- A large χ^2 (small p -value) means observed counts differ from expected by more than chance.
- State degrees of freedom correctly (categories -1 , or $(r-1)(c-1)$).