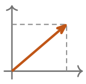
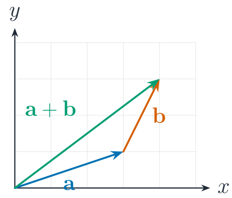
Parametric, Vectors & Matrices

AP Precalculus ▪ Unit 4

AP Precalculus



What we will cover



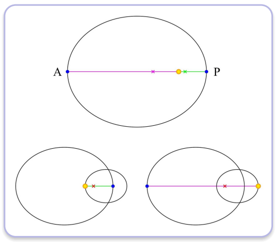
1 Parametric

Parametric functions

A **parametric function** 参数函数 gives x and y each as functions of a **parameter** 参数 t:

$$x = x(t), \qquad y = y(t).$$

> The curve traced need not be a function of x – it may loop or cross itself.



Elliptical orbit

Parametrically Defined Circles and Lines

A **circle** of radius R centered at (h, k) is $x = h + R\cos t,\ y = k + R\sin t$.

> A **line** through (x_0, y_0) with direction (a, b) is $x = x_0 + at,\ y = y_0 + bt$.

Parametrising an implicit curve

what it means	rewriting a curve as $(x(t), y(t))$ – a position at each value of t
why bother	it is easier to graph, and it turns the curve into motion along a path
the circle	$x = r\cos t,\ \ y = r\sin t$ traces $x^2 + y^2 = r^2$ once as t runs 0 to 2π
the ellipse	$x = a\cos t,\ \ y = b\sin t$ – the same idea with different radii
the extra information	a parametrisation also fixes the direction and speed of travel, which the implicit equation does not

> One curve has many parametrisations. Choosing one adds information – where you start, which way you go, how fast – so say what yours does.

Planar motion and rates of change

As t advances, $(x(t), y(t))$ traces the path of a moving point – so t carries direction and timing, which a plain curve cannot show.

> Read each rate separately: $x(t)$ increasing moves right, $y(t)$ decreasing moves down.

Vector coaster

Implicit curves and conic sections

An **implicitly defined** 隐式定义 curve is an equation in x and y – not solved for y. The **conic sections** 圆锥曲线:

- circle $x^2 + y^2 = r^2$
- ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
- parabola $y = ax^2$

> To parametrize an implicit curve, find $x(t)$ and $y(t)$ that satisfy the equation for every t – for a circle, $\cos^2 t + \sin^2 t = 1$ does it automatically.

Implicit equations and the four conics

implicit 隐式	an equation relating x and y without solving for either – $x^2 + y^2 = 25$
the consequence	its graph may fail the vertical-line test, so it need not be a function at all
conic sections 圆锥曲线	circles, ellipses, parabolas and hyperbolas – the curves you get by slicing a cone
each is a quadratic	one general second-degree equation in x and y covers all four
telling them apart	the signs and coefficients of x^2 and y^2: equal – circle; same sign, unequal – ellipse; one missing – parabola; opposite signs – hyperbola

> Complete the square in both variables to get the standard form, and the centre, radius or axes read straight off it.

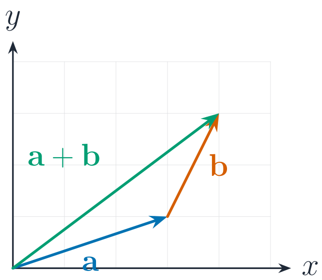
2 Vectors

Vectors

A **vector** 向量 has magnitude and direction, written $\langle a, b\rangle$:

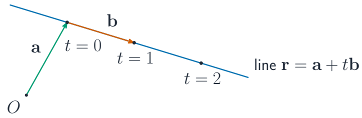
$$|\vec{v}| = \sqrt{a^2 + b^2}, \qquad \theta = \arctan\frac{b}{a}.$$

> Add component-wise; a **scalar** 标量 multiple $k\vec{v}$ scales the length (and flips it if $k < 0$).



Vector-valued functions

A **vector-valued function** 向量值函数 $\vec{p}(t) = \langle x(t), y(t)\rangle$ is a position vector for each t – the same idea as a parametric function, written as one object.



> A line: start at $\vec{a}$, then slide t lots of the direction $\vec{b}$ – $\vec{p}(t) = \vec{a} + t\vec{b}$.

3 Matrices

Matrices, determinant and inverse

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the **determinant** 行列式 is

$$\det A = ad - bc.$$

A is **invertible** 可逆 exactly when $\det A \neq 0$. The inverse undoes the transformation.

$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

$\det A = (3)(4) - (1)(2) = 10 \neq 0$, so A is invertible:

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}.$$

Swap the diagonal, negate the off-diagonal, divide by $\det$.

Matrix arithmetic

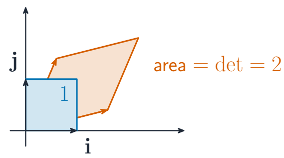
a matrix 矩阵	a rectangular array of numbers
addition	entry by entry, so the two must be the same size
multiplication	combine rows of the first with columns of the second; A's column count must equal B's row count
not commutative	$AB \neq BA$ in general – and often only one of the two is even defined
the identity	I leaves any compatible matrix unchanged: $AI = IA = A$
the inverse	A^{-1} exists only when $\det A \neq 0$, and then $AA^{-1} = I$

As a transformation, $\det A$ is the area scale factor. A zero determinant collapses the plane onto a line, which is exactly why the inverse cannot exist.

Matrices as linear transformations

$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ **stretches the plane**

It sends $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 3 \end{bmatrix}$: $2\times$ horizontally, $3\times$ vertically. Its determinant $2 \times 3 = 6$ means every area is multiplied by 6 – the unit square becomes a 2×3 rectangle.



The determinant is an area scale factor – a negative one flips orientation.

Matrices as functions, and as models

A matrix is a function on vectors: composition of transformations is matrix multiplication, and the identity matrix leaves every vector unchanged.

Order matters: AB and BA usually differ – just like function composition.

a state vector (populations, shares)
↓
× a transition matrix
↓
the next state

A matrix is a function, and it models a system in stages

the claim	a matrix maps input vectors to output vectors, so it is a function on the plane
composition	applying one transformation then another is matrix multiplication – and the order matters
the inverse	A^{-1} undoes A, and exists exactly when $\det A \neq 0$
modelling in stages	matrices model systems that step forward in stages – populations moving between states
a transition matrix	each entry is the proportion moving from one state to another in one step
n steps ahead	multiply by A n times: $\mathbf{v}_n = A^n \mathbf{v}_0$

Because composition is multiplication, "do this transformation n times" becomes "raise the matrix to the nth power" – which is what makes the stage model tractable.

Exam tips

- A **parametric** function gives $x(t)$ and $y(t)$ separately – track the direction of motion as t increases.
- A **vector** has magnitude and direction; add component-by-component and find magnitude with $\sqrt{a^2+b^2}$.
- The **determinant** of a 2×2 matrix is $ad-bc$ (mind the minus sign); it scales area under the transformation.
- A 2×2 matrix transforms the plane (rotate, reflect, stretch); an inverse exists only when the determinant is non-zero.
- Note this unit is not tested on the AP Precalculus Exam (Units 1–3 only), but it underpins later courses.

4 Recap

Unit 4 – key takeaways

1 Parametric

t adds direction and timing a plain curve cannot show

2 Conics

implicit curves; parametrize with $\cos^2 t+\sin^2 t=1$

3 Vectors

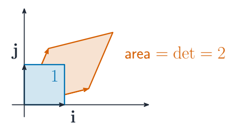
magnitude and direction; add component-wise

4 Matrices

$\det = ad-bc$ is the area scale factor; $\neq 0$ means invertible

Check yourself

1. Parametrize a circle of radius r centred at (h,k).
2. Find $\det\begin{bmatrix}3 & 1\\ 2 & 4\end{bmatrix}$.
3. What does a determinant of 6 do to areas?
4. Give the magnitude of $\langle a,b\rangle$.



Answers **1.** $x=h+r\cos t,\ y=k+r\sin t$ **2.** 10 **3.** multiplies every area by 6 **4.** $\sqrt{a^2+b^2}$