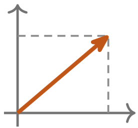
# Parametric, Vectors & Matrices

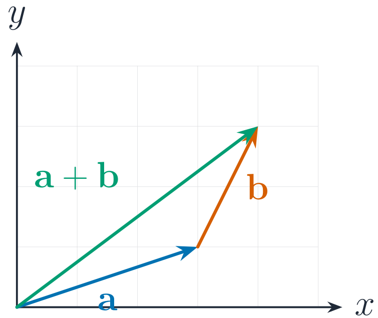
## AP Precalculus ▪ Unit 4

AP Precalculus



# What we will cover

1. Parametric functions
2. Planar motion & rates
3. Circles lines & conics
4. Implicit curves
5. Vectors
6. Matrices & determinants
7. Linear transformations



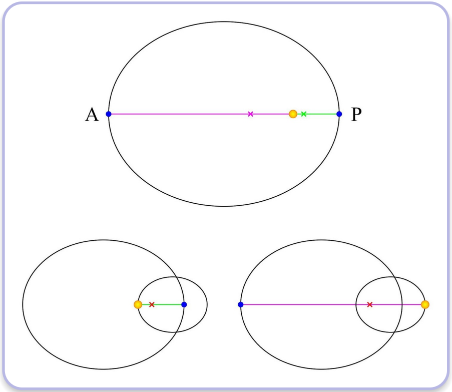
# 1 Parametric

# Parametric functions

A **parametric function** 参数函数 gives $x$ and $y$ each as functions of a **parameter** 参数 $t$:

$$x = x(t), \qquad y = y(t).$$

> The curve traced need not be a function of $x$ – it may loop or cross itself.




*Elliptical orbit*

# Parametrically Defined Circles and Lines

A **circle** of radius $R$ centered at $(h, k)$ is $x = h + R\cos t,\ \ y = k + R\sin t$.

> A **line** through $(x_0, y_0)$ with direction $(a, b)$ is $x = x_0 + at,\ \ y = y_0 + bt$.

# Parametrising an implicit curve

| | |
|---|---|
| **what it means** | rewriting a curve as $\big(x(t), y(t)\big)$ – a position at each value of $t$ |
| **why bother** | it is easier to graph, and it turns the curve into motion along a path |
| **the circle** | $x = r\cos t,\ \ y = r\sin t$ traces $x^2 + y^2 = r^2$ once as $t$ runs $0$ to $2\pi$ |
| **the ellipse** | $x = a\cos t,\ \ y = b\sin t$ – the same idea with different radii |
| **the extra information** | a parametrisation also fixes the direction and speed of travel, which the implicit equation does not |

> One curve has many parametrisations. Choosing one adds information – where you start, which way you go, how fast – so say what yours does.

# Planar motion and rates of change

As $t$ advances, $(x(t), y(t))$ traces the path of a moving point – so $t$ carries direction and timing, which a plain curve cannot show.

> Read each rate separately: $x(t)$ increasing moves right, $y(t)$ decreasing moves down.

*Vector coaster*

# Implicit curves and conic sections

An **implicitly defined** 隐式定义 curve is an equation in $x$ and $y$ – not solved for $y$. The **conic sections** 圆锥曲线:

- circle $x^2 + y^2 = r^2$
- ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
- parabola $y = ax^2$

> To parametrize an implicit curve, find $x(t)$ and $y(t)$ that satisfy the equation for every $t$ – for a circle, $\cos^2 t + \sin^2 t = 1$ does it automatically.

# Implicit equations and the four conics

| | |
|---|---|
| **implicit 隐式** | an equation relating $x$ and $y$ without solving for either – $x^2 + y^2 = 25$ |
| **the consequence** | its graph may fail the vertical-line test, so it need not be a function at all |
| **conic sections 圆锥曲线** | circles, ellipses, parabolas and hyperbolas – the curves you get by slicing a cone |
| **each is a quadratic** | one general second-degree equation in $x$ and $y$ covers all four |
| **telling them apart** | the signs and coefficients of $x^2$ and $y^2$: equal – circle; same sign, unequal – ellipse; one missing – parabola; opposite signs – hyperbola |

Complete the square in both variables to get the standard form, and the centre, radius or axes read straight off it.

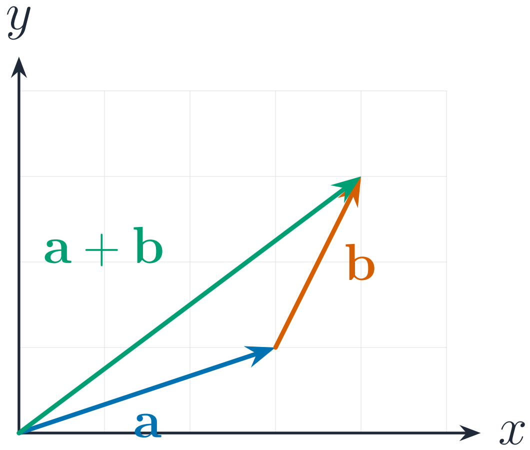
# 2 Vectors

# Vectors

A **vector** 向量 has magnitude and direction, written $\langle a, b \rangle$:

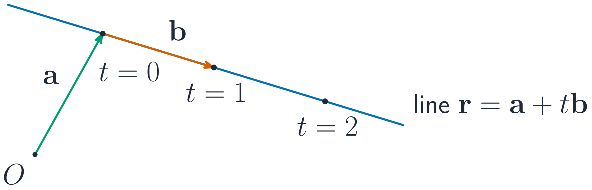
$$|\vec{v}| = \sqrt{a^2 + b^2}, \qquad \theta = \arctan \frac{b}{a}.$$

> Add component-wise; a **scalar** 标量 multiple $k\vec{v}$ scales the length (and flips it if $k < 0$).



# Vector-valued functions

A **vector-valued function** 向量值函数 $\vec{p}(t) = \langle x(t), y(t) \rangle$ is a position vector for each $t$ – the same idea as a parametric function, written as one object.




A line: start at $\vec{a}$, then slide $t$ lots of the direction $\vec{b}$ – $\vec{p}(t) = \vec{a} + t\vec{b}$.

# 3 Matrices

# Matrices, determinant and inverse

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the **determinant** 行列式 is

$$\det A = ad - bc.$$

> $A$ is **invertible** 可逆 exactly when $\det A \neq 0$. The inverse undoes the transformation.

$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

$\det A = (3)(4) - (1)(2) = 10 \neq 0$, so $A$ is invertible:

$$A^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}.$$

Swap the diagonal, negate the off-diagonal, divide by $\det$.

# Matrix arithmetic

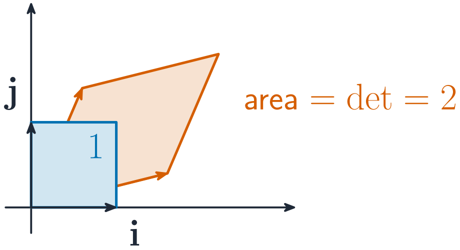
| | |
|---|---|
| **a matrix 矩阵** | a rectangular array of numbers |
| **addition** | entry by entry, so the two must be the same size |
| **multiplication** | combine rows of the first with columns of the second; $A$'s column count must equal $B$'s row count |
| **not commutative** | $AB \neq BA$ in general – and often only one of the two is even defined |
| **the identity** | $I$ leaves any compatible matrix unchanged: $AI = IA = A$ |
| **the inverse** | $A^{-1}$ exists only when $\det A \neq 0$, and then $AA^{-1} = I$ |

As a transformation, $\det A$ is the area scale factor. A zero determinant collapses the plane onto a line, which is exactly why the inverse cannot exist.

# Matrices as linear transformations

**$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ stretches the plane**

It sends $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 3 \end{bmatrix}$: $2\times$ horizontally, $3\times$ vertically. Its determinant $2 \times 3 = 6$ means every area is multiplied by 6 – the unit square becomes a $2 \times 3$ rectangle.

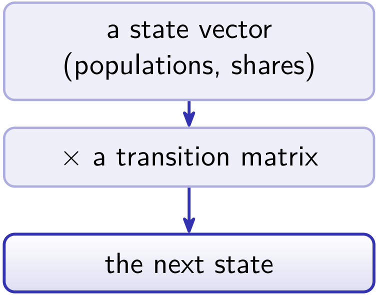



> The determinant is an area scale factor – a negative one flips orientation.

# Matrices as functions, and as models

A matrix is a function on vectors: composition of transformations is matrix multiplication, and the identity matrix leaves every vector unchanged.

> Order matters: $AB$ and $BA$ usually differ – just like function composition.



# A matrix is a function, and it models a system in stages

| | |
|---|---|
| **the claim** | a matrix maps input vectors to output vectors, so it is a function on the plane |
| **composition** | applying one transformation then another is matrix multiplication – and the order matters |
| **the inverse** | $A^{-1}$ undoes $A$, and exists exactly when $\det A \neq 0$ |
| **modelling in stages** | matrices model systems that step forward in stages – populations moving between states |
| **a transition matrix** | each entry is the proportion moving from one state to another in one step |
| ***n* steps ahead** | multiply by $A$ $n$ times: $\mathbf{v}_n = A^n\mathbf{v}_0$ |

> Because composition is multiplication, "do this transformation *n* times" becomes "raise the matrix to the *n*th power" – which is what makes the stage model tractable.

# Exam tips

- A **parametric** function gives $x(t)$ and $y(t)$ separately – track the direction of motion as $t$ increases.
- A **vector** has magnitude and direction; add component-by-component and find magnitude with $\sqrt{a^2 + b^2}$.
- The **determinant** of a $2 \times 2$ matrix is $ad - bc$ (mind the minus sign); it scales area under the transformation.
- A $2 \times 2$ matrix transforms the plane (rotate, reflect, stretch); an inverse exists only when the determinant is non-zero.
- Note this unit is not tested on the AP Precalculus Exam (Units 1–3 only), but it underpins later courses.

# 4 Recap

# Unit 4 – key takeaways

1 **Parametric**

$t$ adds direction and timing a plain curve cannot show

2 **Conics**

implicit curves; parametrize with $\cos^2 t + \sin^2 t = 1$

3 **Vectors**

magnitude and direction; add component-wise

4 **Matrices**

$\det = ad - bc$ is the area scale factor; $\neq 0$ means invertible

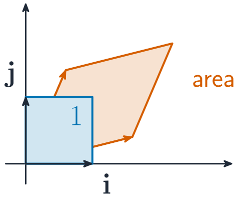
# Check yourself

1. Parametrize a circle of radius $r$ centred at $(h, k)$.

2. Find $\det \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$.

3. What does a determinant of $6$ do to areas?

4. Give the magnitude of $\langle a, b \rangle$.




**Answers** **1.** $x = h + r\cos t, \; y = k + r\sin t$ **2.** $10$ **3.** multiplies every area by $6$ **4.** $\sqrt{a^2 + b^2}$