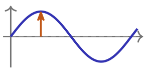
Trigonometric & Polar Functions

AP Precalculus ▪ Unit 3

AP Precalculus



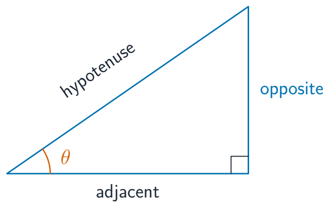
What we will cover

1 Periodic phenomena

Periodic phenomena

A **periodic** 周期 function repeats its output values over equal input intervals – tides, daylight hours, a rotating wheel.

> Period and **frequency** 频率 are reciprocals. $\sin\theta$ has period 2π.



Periodic functions, and angles in standard position

periodic 周期性	the output pattern repeats at regular input steps
the period 周期	the smallest positive k with $f(x+k) = f(x)$ for all x
a cycle 周期循环	one complete repetition of the pattern
standard position 标准位置	vertex at the origin, initial ray along the positive x-axis; positive angles turn anticlockwise
radian 弧度 measure	the arc length on a unit circle – so a full turn is 2π, and π radians is $180°$
why radians	the arc-length and rate formulas only work in radians

> Anything that **oscillates** 振荡 – tides, daylight hours, a wheel – is modelled here, and always by fixing period, amplitude and midline first.

Sine, cosine and tangent on the unit circle

For a point (x, y) at distance r along the terminal ray:

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}, \quad \tan\theta = \frac{y}{x}.$$

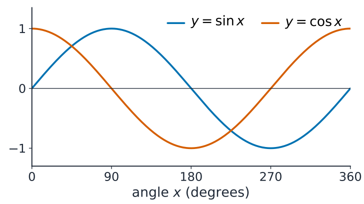
> $\tan\theta$ is the slope of the terminal ray – which is why it is undefined at $\theta = \frac{\pi}{2}$.

Ferris color

2 Sinusoids

Amplitude, midline and period

- **Amplitude** 振幅 = half the distance between max and min
- **Midline** 中线 $y = d$ = the average of max and min
- Period $= \dfrac{2\pi}{b}$



Worked example – read a sinusoid

$f(\theta) = 3\sin(2\theta) + 1$

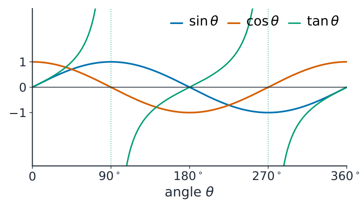
Amplitude 3; period $\dfrac{2\pi}{2} = \pi$; midline $y = 1$. So the maximum is $1 + 3 = 4$ and the minimum is $1 - 3 = -2$.

The general form is

$$a\sin\big(b(\theta + c)\big) + d.$$

> a sets amplitude, b the period, $-c$ the phase shift (a $+c$ shifts left), d the midline.

Transformations and data modelling



To model periodic data (tides, daylight, temperature): read the **period** from where the pattern repeats, get the **amplitude** and **midline** from the max and min, and set the phase shift so a peak...

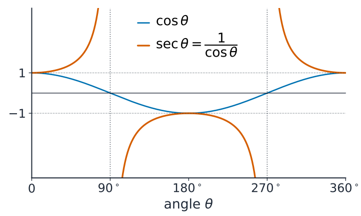
> Technology fits a sinusoidal regression.

3 Inverse trig & equations

Tangent, and the reciprocal functions

$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta}.$$

> $\tan\theta$ has period π (not 2π) and a vertical asymptote wherever $\cos\theta = 0$.



Inverse trigonometric functions

Sine, cosine and tangent are not one-to-one, so each inverse needs a restricted domain.

> An inverse trig function returns an angle – and only the one in its restricted range.

arcsin: range $[-\frac{\pi}{2}, \frac{\pi}{2}]$

arccos: range $[0, \pi]$

arctan: range $(-\frac{\pi}{2}, \frac{\pi}{2})$

Worked example – a trigonometric equation

Solve $2\sin\theta = 1$ on $[0, 2\pi)$

$\sin\theta = \frac{1}{2}$. On the unit circle sine is positive in the first and second quadrants:

$$\theta = \frac{\pi}{6} \quad \text{or} \quad \theta = \frac{5\pi}{6}.$$

Over all reals, add $2\pi k$ to each.

> A trig equation has infinitely many solutions – the calculator gives you one. Add the period, and find the second quadrant solution yourself.

Equivalent representations – the identities

Pythagorean: $\sin^2\theta + \cos^2\theta = 1$

Sum: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Double angle: $\sin 2\theta = 2\sin\theta\cos\theta$

> An **identity** 恒等式 is true for every θ – use one to rewrite an expression into a form you can solve.

The identities the exam expects

Pythagorean identity **毕达哥拉斯恒等式**	$\sin^2\theta + \cos^2\theta = 1$, and its two derived forms with tan and cot
sum identities **和角公式**	$\sin(a \pm b) = \sin a\cos b \pm \cos a\sin b$; $\cos(a \pm b) = \cos a\cos b \mp \sin a\sin b$
double-angle **倍角公式**	$\sin 2\theta = 2\sin\theta\cos\theta$; $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$
arcsine **反正弦** and friends	inverse trig functions need a restricted domain to be functions at all
the special triangles	the **equilateral triangle** 等边三角形 halved gives $30°$–$60°$; the square halved gives $45°$

> Every one of these lets you rewrite an expression into a form you can solve. Identities are tools for rewriting, not facts to recite.

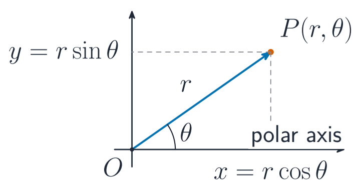
4 Polar

Polar coordinates

Polar coordinates 极坐标 give a point by distance r and angle θ:

$$x = r\cos\theta, \quad y = r\sin\theta,$$

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$

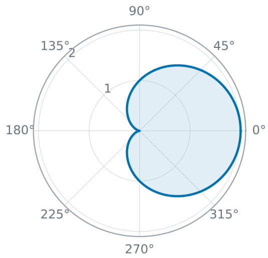


Polar graphs and their rates of change

A polar function $r = f(\theta)$ sweeps a curve as θ turns.

> r increasing means the curve moves away from the origin; r decreasing, toward it. A negative r plots opposite the angle.

$r = 1 + \cos\theta$



Reading a polar graph

what it is	the graph of $r = f(\theta)$ is all points (r, θ) as θ sweeps around
restricting θ	draws only part of the curve – so the domain is part of the answer
r increasing	the curve moves away from the origin as θ grows
r decreasing	it moves toward the origin – and a negative r plots in the opposite direction
the extremes	where r switches between increasing and decreasing, the distance from the origin reaches a relative extreme
average rate	$\frac{\Delta r}{\Delta\theta}$ – how fast the distance changes per radian turned

> Track distance from the origin against angle, separately. Trying to read a polar graph as if it were Cartesian is what makes it confusing.

Periodic Phenomena, continued

> The ISS in orbit: trigonometric and polar functions describe periodic and circular motion

Exam tips

- Use radians and the **unit circle**: $\cos\theta$ and $\sin\theta$ are the coordinates, always between -1 and 1.
- For a sinusoid $a\sin(b(\theta + c)) + d$: $|a|$ is the amplitude, $\frac{2\pi}{|b|}$ the period, d the midline, and the phase shift is $-c$ (a $+c$ shifts left).
- Model periodic data by reading period, amplitude, and midline from the max and min.
- **Inverse trig** functions need a restricted domain and return an angle; trig equations have infinitely many solutions (add the period).
- Convert polar rectangular with $x = r\cos\theta$, $y = r\sin\theta$.

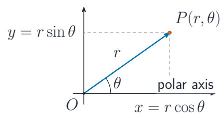
5 Recap

Unit 3 – key takeaways

1. **The unit circle** — $\cos = x/r$, $\sin = y/r$, $\tan =$ the ray's slope
2. **Sinusoids** — $a\sin(b(\theta + c)) + d$: amplitude, period, shift, midline
3. **Equations** — infinitely many solutions – add the period
4. **Polar** — $x = r\cos\theta$, $y = r\sin\theta$; r rising moves away from O

Check yourself

1. Give the amplitude, period and midline of $3\sin(2\theta)+1$.
2. Solve $2\sin\theta = 1$ on $[0, 2\pi)$.
3. Why does $\arcsin$ need a restricted domain?
4. Convert: what are x and y in terms of r and θ?



Answers **1.** 3, π, $y=1$ **2.** $\frac{\pi}{6}$, $\frac{5\pi}{6}$ **3.** sine is not one-to-one **4.** $x = r\cos\theta$, $y = r\sin\theta$