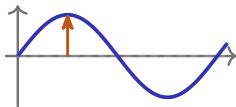


Trigonometric & Polar Functions

AP Precalculus ■ Unit 3

AP Precalculus



What we will cover

- 1 Periodic phenomena
- 2 Sine cosine & tangent
- 3 Sinusoidal functions
- 4 Transformations & modelling
- 5 Inverse trig & equations
- 6 Identities
- 7 Polar functions

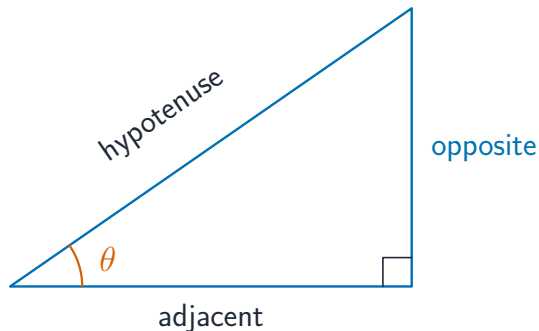
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Periodic phenomena

Periodic phenomena

A **periodic** 周期 function repeats its output values over equal input intervals – tides, daylight hours, a rotating wheel.

Period and **frequency** 频率 are reciprocals. $\sin \theta$ has period 2π .



Periodic functions, and angles in standard position

periodic 周期性

the output pattern repeats at regular input steps

the period 周期

the **smallest** positive k with $f(x + k) = f(x)$ for all x

a cycle 周期循环

one complete repetition of the pattern

standard position 标准位置

vertex at the origin, initial ray along the **positive x-axis**; positive angles turn anticlockwise

radian 弧度 **measure**

the arc length on a **unit** circle – so a full turn is 2π , and π radians is 180°

why radians

the arc-length and rate formulas only work in radians

Anything that **oscillates** 振荡 – tides, daylight hours, a wheel – is modelled here, and always by fixing period, amplitude and midline first.

Sine, cosine and tangent on the unit circle

For a point (x, y) at distance r along the terminal ray:

$$\cos \theta = \frac{x}{r}, \quad \sin \theta = \frac{y}{r}, \quad \tan \theta = \frac{y}{x}.$$

$\tan \theta$ is the slope of the terminal ray –
which is why it is undefined at $\theta = \frac{\pi}{2}$.



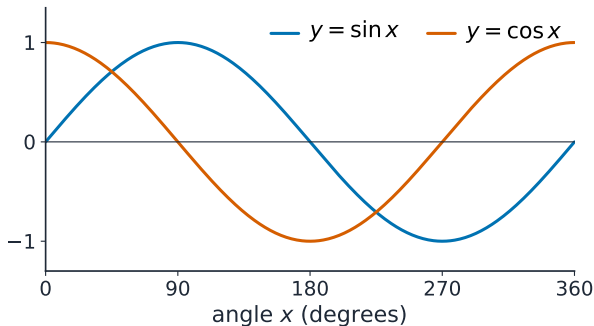
Ferris color

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Sinusoids

Amplitude, midline and period

- **Amplitude** 振幅 = half the distance between max and min
- **Midline** 中线 $y = d$ = the average of max and min
- **Period** $= \frac{2\pi}{b}$



Worked example – read a sinusoid

$$f(\theta) = 3 \sin(2\theta) + 1$$

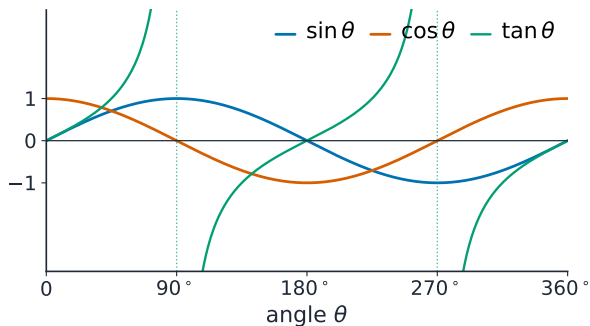
Amplitude 3 ; period $\frac{2\pi}{2} = \pi$; midline $y = 1$. So the maximum is $1 + 3 = 4$ and the minimum is $1 - 3 = -2$.

The general form is

$$a \sin(b(\theta + c)) + d.$$

a sets amplitude, b the period, $-c$ the **phase shift** ($a + c$ shifts left), d the midline.

Transformations and data modelling



To model periodic data (tides, daylight, temperature): read the **period** from where the pattern repeats, get the **amplitude** and **midline** from the max and min, and set the phase shift so a peak...

Technology fits a sinusoidal regression.

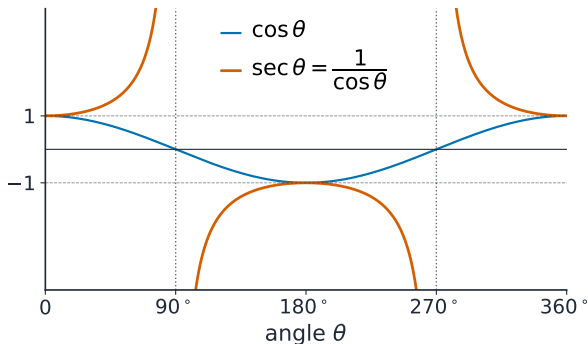
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Inverse trig & equations

Tangent, and the reciprocal functions

$$\sec \theta = \frac{1}{\cos \theta}, \quad \csc \theta = \frac{1}{\sin \theta}, \quad \cot \theta = \frac{1}{\tan \theta}.$$

$\tan \theta$ has period π (not 2π) and a **vertical asymptote** wherever $\cos \theta = 0$.



Inverse trigonometric functions

Sine, cosine and tangent are **not** one-to-one, so each inverse needs a **restricted domain**.

An inverse trig function **returns an angle** – and only the one in its restricted range.

arcsin: range $[-\frac{\pi}{2}, \frac{\pi}{2}]$

arccos: range $[0, \pi]$

arctan: range $(-\frac{\pi}{2}, \frac{\pi}{2})$

Worked example – a trigonometric equation

Solve $2 \sin \theta = 1$ **on** $[0, 2\pi)$

$\sin \theta = \frac{1}{2}$. On the unit circle sine is positive in the **first and second** quadrants:

$$\theta = \frac{\pi}{6} \quad \text{or} \quad \theta = \frac{5\pi}{6}.$$

Over all reals, add $2\pi k$ to each.

A trig equation has **infinitely many** solutions – the calculator gives you **one**. Add the period, and find the **second** quadrant solution yourself.

Equivalent representations – the identities

Pythagorean: $\sin^2 \theta + \cos^2 \theta = 1$

Sum: $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

Double angle: $\sin 2\theta = 2 \sin \theta \cos \theta$

An **identity** 恒等式 is true for **every** θ – use one to rewrite an expression into a form you can solve.

The identities the exam expects

Pythagorean identity 毕达哥拉斯恒等式

$\sin^2 \theta + \cos^2 \theta = 1$, and its two derived forms with $\tan$ and $\cot$

sum identities 和角公式

$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$; $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

double-angle 倍角公式

$\sin 2\theta = 2 \sin \theta \cos \theta$; $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 1 - 2 \sin^2 \theta$

arcsine 反正弦 and friends

inverse trig functions need a **restricted domain** to be functions at all

the special triangles

the **equilateral triangle** 等边三角形 halved gives $30^\circ - 60^\circ$; the square halved gives 45°

Every one of these lets you rewrite an expression into a form you can solve. Identities are **tools for rewriting**, not facts to recite.

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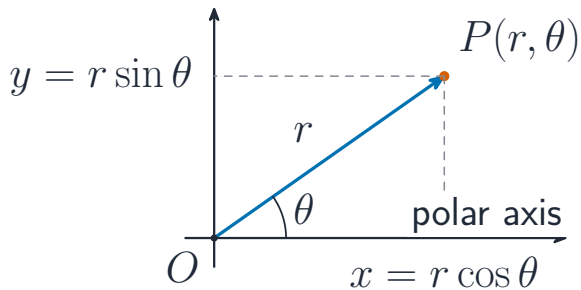
Polar

Polar coordinates

Polar coordinates 极坐标 give a point by distance r and angle θ :

$$x = r \cos \theta, \quad y = r \sin \theta,$$

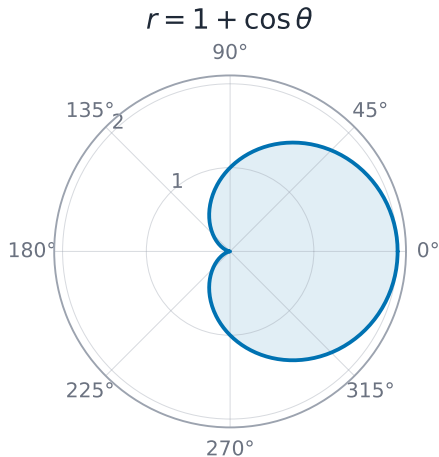
$$r = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x}.$$



Polar graphs and their rates of change

A polar function $r = f(\theta)$ sweeps a curve as θ turns.

r increasing means the curve moves away from the origin; r decreasing, toward it. A negative r plots opposite the angle.



Reading a polar graph

what it is

the graph of $r = f(\theta)$ is all points (r, θ) as θ sweeps around

restricting θ

draws only **part** of the curve – so the domain is part of the answer

 r increasing

the curve moves **away** from the origin as θ grows

 r decreasing

it moves toward the origin – and a **negative** r plots in the opposite direction

the extremes

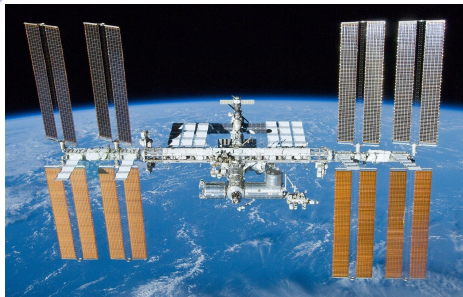
where r switches between increasing and decreasing, the distance from the origin reaches a relative extreme

average rate

$\frac{\Delta r}{\Delta \theta}$ – how fast the distance changes per radian turned

Track **distance from the origin** against **angle**, separately. Trying to read a polar graph as if it were Cartesian is what makes it confusing.

Periodic Phenomena, continued



The ISS in orbit: trigonometric and polar functions describe periodic and circular motion

Exam tips

- Use radians and the **unit circle**: $\cos \theta$ and $\sin \theta$ are the coordinates, always between -1 and 1 .
- For a sinusoid $a \sin(b(\theta + c)) + d$: $|a|$ is the amplitude, $\frac{2\pi}{|b|}$ the period, d the midline, and the phase shift is $-c$ ($a + c$ shifts left).
- Model periodic data by reading period, amplitude, and midline from the max and min.
- **Inverse trig** functions need a restricted domain and return an angle; trig equations have infinitely many solutions (add the period).
- Convert polar rectangular with $x = r \cos \theta$, $y = r \sin \theta$.

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Recap

Unit 3 – key takeaways

1

The unit circle

$\cos = x/r$, $\sin = y/r$, $\tan =$ the ray's slope

2

Sinusoids

$a \sin(b(\theta + c)) + d$: amplitude, period, shift, midline

3

Equations

infinitely many solutions – add the period

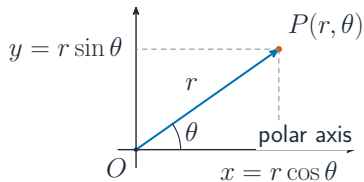
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Polar

$x = r \cos \theta$, $y = r \sin \theta$; r rising moves away from O

Check yourself

- 1 Give the amplitude, period and midline of $3 \sin(2\theta) + 1$.
- 2 Solve $2 \sin \theta = 1$ on $[0, 2\pi)$.
- 3 Why does $\arcsin$ need a restricted domain?
- 4 Convert: what are x and y in terms of r and θ ?



Answers 1. 3, π , $y = 1$ 2. $\frac{\pi}{6}$, $\frac{5\pi}{6}$ 3. sine is not one-to-one 4. $x = r \cos \theta$, $y = r \sin \theta$