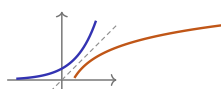


Exponential & Logarithmic Functions

AP Precalculus • Unit 2

AP Precalculus



Exponential & Logarithmic Functions

What we will cover

- 1 Arithmetic & geometric sequences
- 2 Linear vs exponential change
- 3 Exponential functions
- 4 Composition & inverses
- 5 Logarithms
- 6 Solving equations
- 7 Modelling & semi-log plots

1 Sequences

Exponential & Logarithmic Functions

Sequences

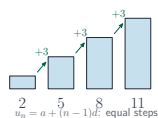
Arithmetic and geometric sequences

A **sequence** 数列 maps whole numbers to reals – so its graph is **discrete points**, not a curve.

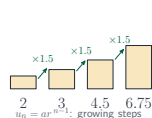
- **arithmetic** 等差数列: **common difference** 公差 d , $a_n = a_0 + dn$
- **geometric** 等比数列: **common ratio** 公比 r , $g_n = g_0 r^n$

Arithmetic and geometric sequences, in detail

arithmetic: add d each step



geometric: multiply by r each step



Arithmetic grows by the **same amount** each step; geometric by a **larger amount** each step.

A **sequence** 数列 maps whole numbers to reals – so its graph is **discrete points**, not a curve.

Exponential & Logarithmic Functions

Sequences

Worked example – addition versus multiplication

Compare the fourth terms

Arithmetic, $a_0 = 3$, $d = 5$:

$$a_4 = 3 + 5(4) = 23$$

Geometric, $g_0 = 2$, $r = 3$:

$$g_4 = 2 \cdot 3^4 = 162$$

Multiplication makes the geometric one **pull far ahead**.

The continuous cousins: **linear** $f(x) = b + mx$ mirrors arithmetic (repeated **addition**); **exponential** $f(x) = ab^x$ mirrors geometric (repeated **multiplication**).

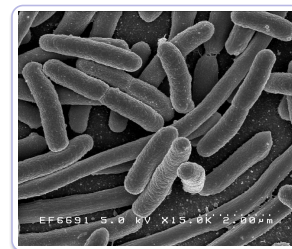
2 Exponentials

Exponential functions

$$f(x) = ab^x, \quad a \neq 0, b > 0, b \neq 1$$

initial value 初始值 a , **base** 底数 b ;
domain: all reals.

- $b > 1$: **growth** 指数增长
- $0 < b < 1$: **decay** 指数衰减



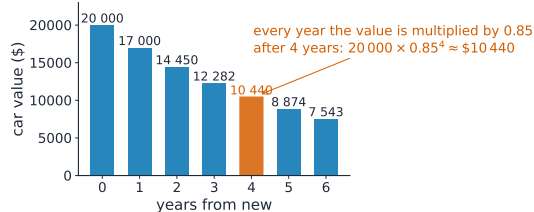
Bacteria exponential

What makes a function exponential

the defining property	outputs are proportional over equal-length input intervals – a constant <i>ratio</i> , where a linear function has a constant <i>difference</i>
the consequence	an exponential is always increasing or always decreasing, and always concave up or always concave down
exponential growth 指数增长	$b > 1$: the value multiplies each step
exponential decay	$0 < b < 1$: it multiplies by less than one, approaching zero without reaching it
in data	sometimes a constant must be added to the data before the proportional pattern shows – a shifted asymptote hides it

Test data by **dividing** consecutive outputs, not subtracting. A constant ratio is exponential; a constant difference is linear.

Exponential decay



Exponential decay: a fixed percentage is lost each period

Exponent rules are graph transformations

Product $b^m b^n = b^{m+n}$
a horizontal **shift** b^{x+k}
is a vertical **stretch**

Power $(b^m)^n = b^{mn}$
a horizontal **stretch** b^{cx}
is a **base change** $(b^c)^x$

Negative $b^{-n} = \frac{1}{b^n}$

Unit fraction $b^{1/k} = \sqrt[k]{b}$
(the k th **root** 方根)

Exponent rules are graph transformations, in detail

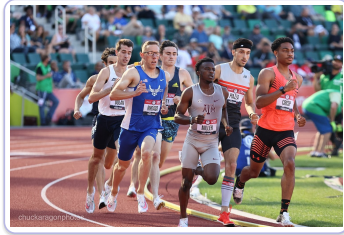


Composition matryoshka

Modelling, and choosing between models

Exponentials model growth by a **constant proportion** over equal intervals.

- build from a ratio + an initial value, or from **two points**
- **exponential regression** 指数回归 fits a data set
- the **natural base** 自然底数 $e \approx 2.718$ is the standard



Exponential sprint

Choosing between competing models

the problem	when a rate of change shifts only slightly, linear, quadratic and exponential can all seem to fit
residual 残差	actual minus predicted – the gap the model leaves at each point
residual plot 残差图	a model is appropriate when its residual plot shows no pattern
a pattern means	the model is missing structure the data has – a curve in the residuals means you fitted a line to a curve
the error	the gap between predicted and actual; context decides whether over- or under-estimating is safer

A high R^2 with a patterned residual plot is still the wrong model. The residual plot is the check that catches what the summary number hides.

3 Composition & inverses

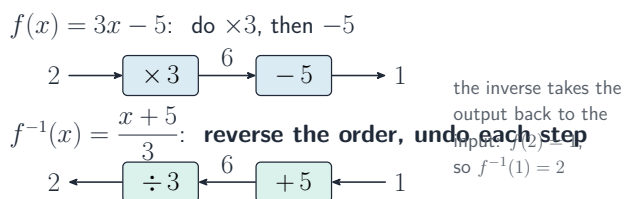
Composition of functions

The **composite** 复合函数 $(f \circ g)(x) = f(g(x))$ feeds g 's output into f .

Composition is **not commutative**: $f(g(x))$ and $g(f(x))$ usually differ.

To build it, **substitute** $g(x)$ for every x in f . The **identity** 恒等函数 $f(x) = x$ changes nothing.

Composition of functions, in detail



The **composite** 复合函数 $(f \circ g)(x) = f(g(x))$ feeds g 's output into f .

Composing, decomposing, and inverting

composite function 复合函数	$(f \circ g)(x) = f(g(x))$ – inner function first, and the order matters
the domain	x must be allowed in g , and $g(x)$ must be allowed in f
decomposing 分解	splitting a function into simpler pieces – an additive shift is composing with $x + k$, a dilation is composing with bx
finding an inverse	swap x and y in $y = f(x)$, then solve for y
the check	$f(f^{-1}(x)) = x$: composing a function with its inverse gives the identity function 恒等函数
the restriction	context, or a failed horizontal-line test, may limit where the inverse applies

Decomposition is why transformations and composition are the same topic: every shift and stretch is a composition with a simple function.

Inverse functions

f is **invertible** 可逆 where each output comes from a **unique** input. Then $f(a) = b \Rightarrow f^{-1}(b) = a$, and

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x.$$

Domain and range **swap**; the graph reflects over $y = x$.

swap x and y in $y = f(x)$

solve for y

You may

restrict the domain to force invertibility.

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Logarithms

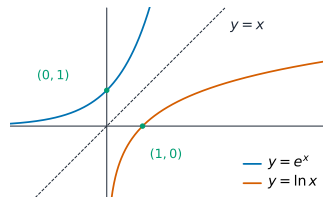
The logarithm answers "what exponent?"

$$\log_b c = a \iff b^a = c$$

The **logarithm** 对数 is the **inverse** of the exponential, so

$$\log_b(b^x) = x, \quad b^{\log_b x} = x.$$

$\ln x = \log_e x$ is the **natural logarithm** 自然对数, inverse of e^x .



The logarithmic function, and its properties

$f(x) = \log_b x$: domain $x > 0$, range all reals, a **vertical asymptote** at $x = 0$ – the mirror of the exponential's horizontal one.

It grows **very slowly** for large x .

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

$$\log_b(x^n) = n \log_b x$$

change of base 换底: $\log_b x = \frac{\ln x}{\ln b}$

Solving exponential and logarithmic equations

Use the fact that they are **inverses**:

- isolate the exponential, then **take a log** of both sides
- isolate the log, then **exponentiate** both sides

Check for **extraneous solutions** – a log's argument must stay **positive**.

Solve $2 \cdot 3^x = 54$

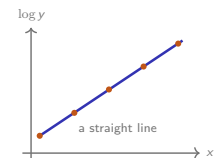
Divide by 2: $3^x = 27 = 3^3$, so $x = 3$.
When the sides are not tidy powers, take logs:

$$5^x = 20 \Rightarrow x = \frac{\ln 20}{\ln 5} \approx 1.86.$$

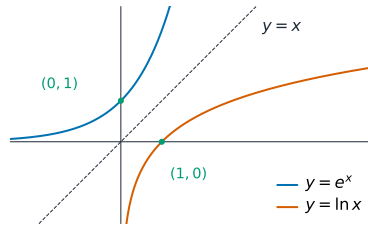
Semi-log plots

On a **logarithmic scale**, each unit is a **multiplicative step** ($10^0, 10^1, 10^2, \dots$).

Plot $\log y$ against x : an **exponential becomes a straight line**, whose slope and intercept recover the model.



Check yourself, in detail



e^x and $\ln x$ are reflections of each other in the line $y = x$

Exam tips

- Distinguish **arithmetic** (add a common difference) from **geometric** (multiply a common ratio) sequences and their linear/exponential function cousins.
- Exponential = repeated **multiplication**, so it eventually outgrows any linear or polynomial model.
- The **logarithm is the inverse** of the exponential ($\log_b c = a \Leftrightarrow b^a = c$); use it to solve $b^x = k$.
- Apply the log laws (product, quotient, power) and the change-of-base formula; the argument of a log must be **positive**.
- On a **semi-log plot** an exponential model becomes a straight line.

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Recap

Unit 2 – key takeaways

1 Two kinds of change

linear repeats addition;
exponential repeats multiplication

2 Exponentials

$f = ab^x$: no extrema, no
inflection, proportional growth

3 Inverses

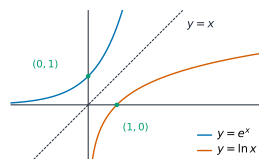
swap domain and range; reflect
over $y = x$

4 Logarithms

the inverse of b^x ; log rules reverse
the exponent rules

Check yourself

- 1 Solve $2 \cdot 3^x = 54$.
- 2 What does $\log_b c = a$ mean, as an exponential?
- 3 Does an exponential function have inflection points?
- 4 On a semi-log plot, what shape is an exponential?



Answers 1. $x = 3$ 2. $b^a = c$ 3. no 4. a straight line