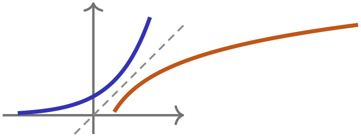
Exponential & Logarithmic Functions

AP Precalculus ▪ Unit 2

AP Precalculus



What we will cover

1. Arithmetic & geometric sequences
2. Linear vs exponential change
3. Exponential functions
4. Composition & inverses
5. Logarithms
6. Solving equations
7. Modelling & semi-log plots

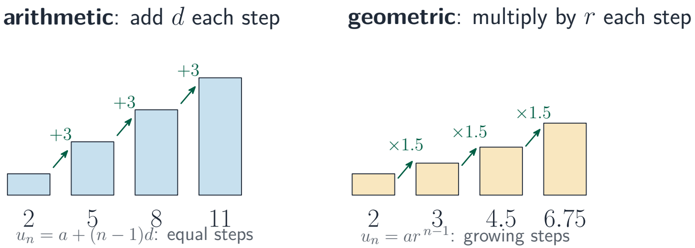
1 Sequences

Arithmetic and geometric sequences

A **sequence** 数列 maps whole numbers to reals – so its graph is discrete points, not a curve.

- **arithmetic** 等差数列: **common difference** 公差 d, $a_n = a_0 + dn$
- **geometric** 等比数列: **common ratio** 公比 r, $g_n = g_0 r^n$

Arithmetic and geometric sequences, in detail



> Arithmetic grows by the same amount each step; geometric by a larger amount each step.

> A **sequence** 数列 maps whole numbers to reals – so its graph is discrete points, not a curve.

Worked example – addition versus multiplication

Compare the fourth terms

Arithmetic, $a_0 = 3$, $d = 5$:

$$a_4 = 3 + 5(4) = 23$$

Geometric, $g_0 = 2$, $r = 3$:

$$g_4 = 2 \cdot 3^4 = 162$$

Multiplication makes the geometric one pull far ahead.

The continuous cousins: linear $f(x) = b + mx$ mirrors arithmetic (repeated addition); exponential $f(x) = ab^x$ mirrors geometric (repeated multiplication).

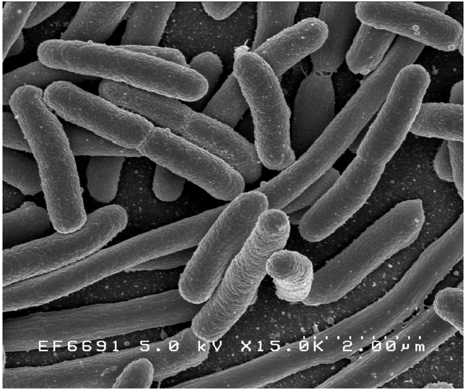
2 Exponentials

Exponential functions

$$f(x) = ab^x, \quad a \neq 0,\ b > 0,\ b \neq 1$$

initial value 初始值 a, **base** 底数 b;
domain: all reals.

- $b > 1$: **growth** 指数增长
- $0 < b < 1$: **decay** 指数衰减



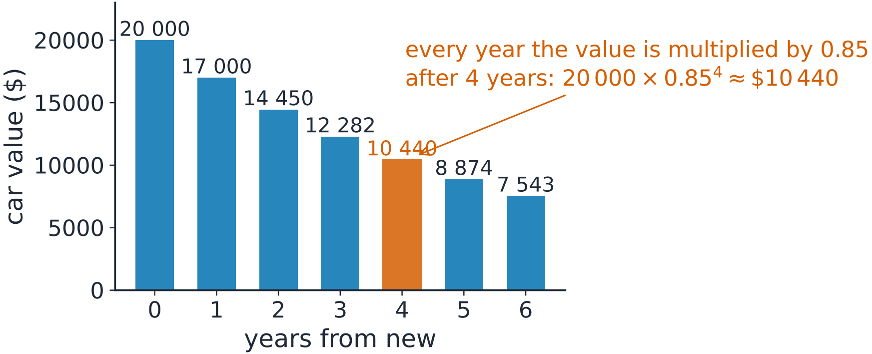
Bacteria exponential

What makes a function exponential

the defining property	outputs are proportional over equal-length input intervals – a constant *ratio*, where a linear function has a constant difference
the consequence	an exponential is always increasing or always decreasing, and always concave up or always concave down
exponential growth 指数增长	$b > 1$: the value multiplies each step
exponential decay	$0 < b < 1$: it multiplies by less than one, approaching zero without reaching it
in data	sometimes a constant must be added to the data before the proportional pattern shows – a shifted asymptote hides it

> Test data by dividing consecutive outputs, not subtracting. A constant ratio is exponential; a constant difference is linear.

Exponential decay



Exponential decay: a fixed percentage is lost each period

Exponent rules are graph transformations

Product $b^m b^n = b^{m+n}$
a horizontal shift b^{x+k}
is a vertical stretch

Power $(b^m)^n = b^{mn}$
a horizontal stretch b^{cx}
is a base change $(b^c)^x$

Negative $b^{-n} = \frac{1}{b^n}$

Unit fraction $b^{1/k} = \sqrt[k]{b}$
(the kth **root** 方根)

Exponent rules are graph transformations, in detail

Composition matryoshka

Modelling, and choosing between models

Exponentials model growth by a constant proportion over equal intervals.

- build from a ratio + an initial value, or from two points
- **exponential regression** 指数回归 fits a data set
- the **natural base** 自然底数 $e \approx 2.718$ is the standard

Exponential sprint

Choosing between competing models

the problem	when a rate of change shifts only slightly, linear, quadratic and exponential can all seem to fit
residual 残差	actual minus predicted – the gap the model leaves at each point
residual plot 残差图	a model is appropriate when its residual plot shows no pattern
a pattern means	the model is missing structure the data has – a curve in the residuals means you fitted a line to a curve
the error	the gap between predicted and actual; context decides whether over- or under-estimating is safer

A high R^2 with a patterned residual plot is still the wrong model. The residual plot is the check that catches what the summary number hides.

3 Composition & inverses

Composition of functions

The **composite** 复合函数 $(f \circ g)(x) = f(g(x))$ feeds g's output into f.

> Composition is not commutative: $f(g(x))$ and $g(f(x))$ usually differ.

To build it, substitute $g(x)$ for every x in f. The **identity** 恒等函数 $f(x) = x$ changes nothing.

Composition of functions, in detail

$f(x) = 3x - 5$: do $\times 3$, then -5

$$2 \longrightarrow \boxed{\times 3} \xrightarrow{6} \boxed{-5} \longrightarrow 1$$

$f^{-1}(x) = \dfrac{x+5}{3}$: **reverse the order, undo each step**

$$2 \longleftarrow \boxed{\div 3} \xleftarrow{6} \boxed{+5} \longleftarrow 1$$

the inverse takes the output back to the input: $f(2) = 1$, so $f^{-1}(1) = 2$

> The **composite** 复合函数 $(f \circ g)(x) = f(g(x))$ feeds g's output into f.

Composing, decomposing, and inverting

composite function 复合函数	$(f \circ g)(x) = f(g(x))$ – inner function first, and the order matters
the domain	x must be allowed in g, and $g(x)$ must be allowed in f
decomposing 分解	splitting a function into simpler pieces – an additive shift is composing with $x + k$, a dilation is composing with bx
finding an inverse	swap x and y in $y = f(x)$, then solve for y
the check	$f(f^{-1}(x)) = x$: composing a function with its inverse gives the **identity function** 恒等函数
the restriction	context, or a failed horizontal-line test, may limit where the inverse applies

Decomposition is why transformations and composition are the same topic: every shift and stretch is a composition with a simple function.

Inverse functions

f is **invertible** 可逆 where each output comes from a unique input. Then $f(a) = b \Rightarrow f^{-1}(b) = a$, and

$$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x.$$

> Domain and range swap; the graph reflects over $y = x$.

$$\boxed{\text{swap } x \text{ and } y \text{ in } y = f(x)} \;\downarrow\; \boxed{\text{solve for } y}$$

You may restrict the domain to force invertibility.

4 Logarithms

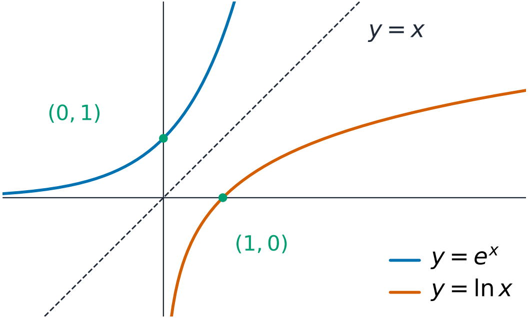
The logarithm answers “what exponent?”

$$\log_b c = a \iff b^a = c$$

The **logarithm** 对数 is the inverse of the exponential, so

$$\log_b(b^x) = x, \qquad b^{\log_b x} = x.$$

> $\ln x = \log_e x$ is the **natural logarithm** 自然对数, inverse of e^x.



The logarithmic function, and its properties

$f(x) = \log_b x$: domain $x > 0$, range all reals, a vertical asymptote at $x = 0$ – the mirror of the exponential's horizontal one.

> It grows very slowly for large x.

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

$$\log_b(x^n) = n \log_b x$$

change of base 换底: $\log_b x = \frac{\ln x}{\ln b}$

Solving exponential and logarithmic equations

Use the fact that they are inverses:

- isolate the exponential, then take a log of both sides
- isolate the log, then exponentiate both sides

> Check for extraneous solutions – a log's argument must stay positive.

Solve $2 \cdot 3^x = 54$

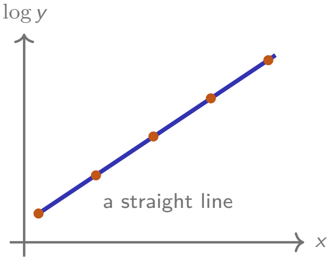
Divide by 2: $3^x = 27 = 3^3$, so $x = 3$.
When the sides are not tidy powers, take logs:

$$5^x = 20 \Rightarrow x = \frac{\ln 20}{\ln 5} \approx 1.86.$$

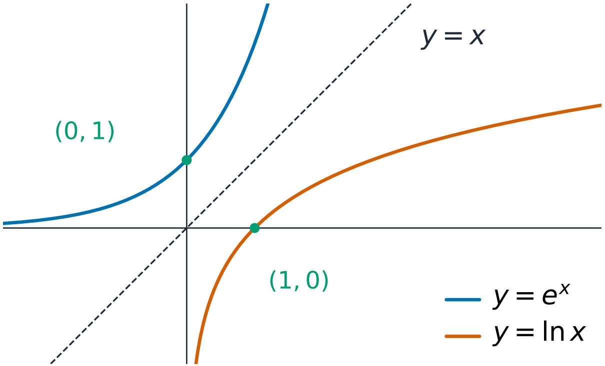
Semi-log plots

On a logarithmic scale, each unit is a multiplicative step $(10^0, 10^1, 10^2, \ldots)$.

> Plot $\log y$ against x: an exponential becomes a straight line, whose slope and intercept recover the model.



Check yourself, in detail



e^x and ln x are reflections of each other in the line y = x

Exam tips

- Distinguish **arithmetic** (add a common difference) from **geometric** (multiply a common ratio) sequences and their linear/exponential function cousins.
- Exponential = repeated **multiplication**, so it eventually outgrows any linear or polynomial model.
- The **logarithm is the inverse** of the exponential ($\log_b c = a \Leftrightarrow b^a = c$); use it to solve $b^x = k$.
- Apply the log laws (product, quotient, power) and the change-of-base formula; the argument of a log must be **positive**.
- On a **semi-log plot** an exponential model becomes a straight line.

5 Recap

Unit 2 – key takeaways

1 Two kinds of change

linear repeats addition; exponential repeats multiplication

2 Exponentials

$f = ab^x$: no extrema, no inflection, proportional growth

3 Inverses

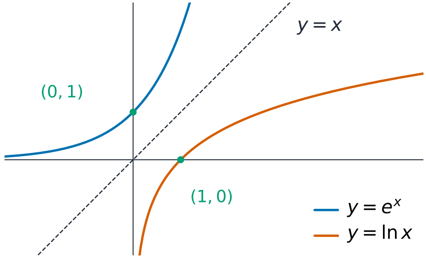
swap domain and range; reflect over $y = x$

4 Logarithms

the inverse of b^x; log rules reverse the exponent rules

Check yourself

1. Solve $2 \cdot 3^x = 54$.
2. What does $\log_b c = a$ mean, as an exponential?
3. Does an exponential function have inflection points?
4. On a semi-log plot, what shape is an exponential?



Answers **1.** $x = 3$ **2.** $b^a = c$ **3.** no **4.** a straight line