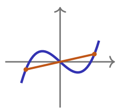
Polynomial & Rational Functions

AP Precalculus ▪ Unit 1

AP Precalculus



What we will cover

1 Change in tandem

Functions, and change in tandem

A **function** 函数 maps each input to exactly one output. Inputs form the **domain** 定义域; outputs the **range** 值域.

- **increasing** 递增: $a < b \Rightarrow f(a) < f(b)$
- **decreasing** 递减: $a < b \Rightarrow f(a) > f(b)$

Bridge curve

The vocabulary every function question uses

a function 函数	a rule mapping each input to exactly one output
domain 定义域 and range 值域	the allowed inputs, and the outputs they produce
independent 自变量 / dependent 因变量	the input you choose, and the output it determines
local extremum 局部极值	a value larger (or smaller) than everything nearby
global maximum 全局最大值	larger than everything in the whole domain – a local one need not be global
piecewise-defined 分段函数	different rules over non-overlapping intervals; pick the branch whose interval contains your input

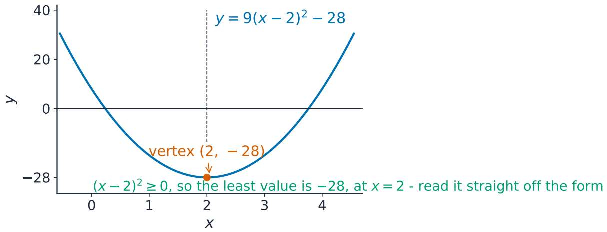
> "Exactly one output" is the whole definition, and it is what the vertical-line test checks on a graph.

The words a function question uses

Independent and dependent	The **independent variable** is the input, plotted horizontally; the **dependent variable** is the output that follows.
Global (absolute) maximum	A **global (absolute) maximum** is the largest output anywhere in the domain; a **local (relative) extremum** is largest or smallest only nearby.
Piecewise-defined function	A **piecewise-defined function** uses a different rule on each interval – so check the endpoints, where the rules meet.
Increasing and decreasing	Say over *which interval*: a function is rarely increasing everywhere, and the interval is part of the answer.

> Describe a graph in this order: domain and range, increase and decrease, extrema, end behaviour.

Completing the square gives the vertex of a parabola



> Completing the square gives the vertex of a parabola

Average rate of change

The **average rate of change** 平均变化率 over $[a, b]$ is the slope of the secant line 割线:

$$\text{avg rate} = \frac{f(b) - f(a)}{b - a}.$$

> Positive rate: both quantities move the same way. Negative: opposite ways.

$f(x) = x^2$

Over $[1, 4]$: $\dfrac{16-1}{3} = 5$.

Over $[1, 2]$: $\dfrac{4-1}{1} = 3$. The rate itself changes – which is exactly why f is not linear.

The rate of the rate

Linear 线性
average rate is constant – so the rate of the rate is zero

Quadratic 二次
rates over equal intervals form a linear pattern – so they change at a constant rate

Parabola bridge

2 Polynomials

Polynomials – degree, leading term, extrema

$$p(x) = a_n x^n + \cdots + a_1 x + a_0, \quad a_n \neq 0$$

Degree 次数 n; **leading term** 首项 $a_n x^n$; **leading coefficient** 首项系数 a_n.

- where it switches increasing/decreasing: a **local extremum** 局部极值
- between two distinct real zeros there is at least one extremum
- an even-degree polynomial has a global max or a global min

> A **point of inflection** 拐点 is where concavity changes – **concave up** 上凹 to **concave down** 下凹 or back.

That is where the rate of change switches between increasing and decreasing.

Symmetry, degree and end behaviour

even 偶函数	$f(-x) = f(x)$ – symmetric across the y-axis, like x^2 and x^4
odd 奇函数	$f(-x) = -f(x)$ – symmetric about the origin, like x and x^3
successive differ-ences 逐次差分	the degree is the least n for which the nth differences of equally spaced outputs become constant
end behavior 末端行为	where the output heads as $x \to \pm\infty$: set by the leading term alone
the two cases	even degree – both ends go the same way; odd degree – opposite ways. The leading coefficient's sign decides which

> Every other term becomes irrelevant far from the origin. That is why end behaviour needs only the degree and the leading coefficient.

Zeros, multiplicity and complex roots

a is a **zero** 零点 when $p(a) = 0$; then $(x - a)$ is a factor. Repeated n times ⇒ **multiplicity** 重数 n.

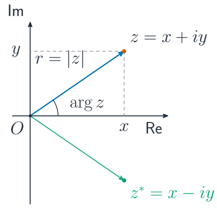
> Counting multiplicities, a degree-n polynomial has exactly n **complex** 复数 zeros.

Non-real zeros come in **conjugate** 共轭 pairs: $a + bi$ and $a - bi$.

Even multiplicity
the graph is tangent – it touches, does not cross

Odd multiplicity
the graph crosses the axis

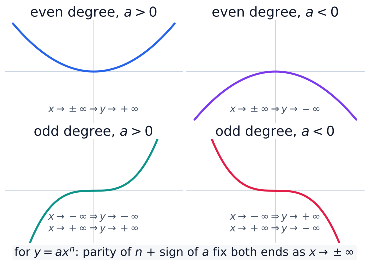
A complex number on an Argand diagram



> A complex number on an Argand diagram: modulus is the distance, argument the angle

End behaviour is decided by the leading term

For large $|x|$, the leading term $a_n x^n$ **dominates** 主导 everything else – so the sign of a_n and the parity of n fix both ends:



3 Rational functions

Rational end behaviour – compare the degrees

top > bottom
follows the quotient polynomial;
if linear, a **slant asymptote** 斜渐近线

equal degrees
horizontal asymptote 水平渐近线
at the ratio of leading coefficients

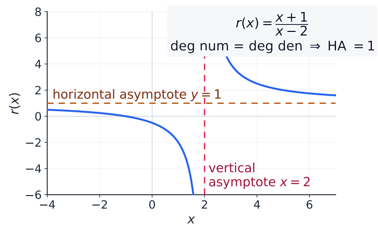
top < bottom
the quotient
$\to 0$, so $y = 0$

> At a horizontal asymptote $y = b$, the outputs get and stay arbitrarily close to b.

Vertical asymptotes and holes

- a **vertical asymptote** 垂直渐近线 at $x = a$: a is a zero of the denominator that does not cancel
- a **hole** 空洞 at $x = c$: the factor cancels – a single missing point at (c, L)

> Check each side of an asymptote – the two sides can go opposite ways.



Worked example – describe a rational function

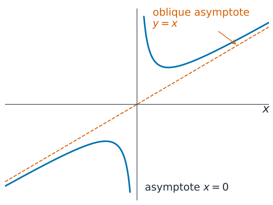
$r(x) = \dfrac{2x^2+3}{x^2-1}$

Equal degrees, so the horizontal asymptote is

$$y = \frac{2}{1} = 2.$$

$x^2 - 1 = 0$ at $x = \pm 1$, and neither cancels, so there are vertical asymptotes at

$$x = 1 \quad \text{and} \quad x = -1.$$



> Top degree one more than the bottom: long division gives a slant asymptote.

Equivalent forms reveal different features

Factored form

shows real zeros – and hence intercepts, holes, vertical asymptotes, domain and range

Standard form

shows the degree and leading term – and hence end behaviour

> **Long division** 多项式长除法 rewrites $f = gq + r$; the **quotient** 商 q is the slant asymptote when the degrees differ by one.

The same expression, written to reveal different things

factored form	shows the zeros immediately
standard form	shows the degree and the leading coefficient, so the end behaviour
vertex form	shows the vertex of a quadratic without any calculation
after polynomial long division 多项式长除法	shows a rational function's asymptote as the quotient, plus a remainder that vanishes
the habit	before calculating, ask which form already *contains* the answer

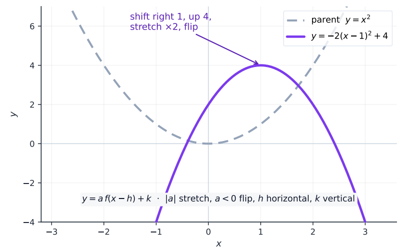
> Rewriting is not busywork; it is choosing the representation in which the question is already answered.

4 Transformations

Transformations of functions

- **Translations** 平移: $f(x) + k$ up/down; $f(x-h)$ right/left
- **Dilations** 伸缩: $a\,f(x)$ vertical; $f(bx)$ horizontal
- **Reflections** 反射: $-f(x)$ over the x-axis; $f(-x)$ over the y-axis

Transformations of functions, in detail



> Transforming the parent parabola: shift, stretch, and reflect

The four transformations, and the direction each moves

$f(x) + k$	vertical shift up by k – as expected
$f(x + k)$	horizontal shift left by k – the *opposite* of what the sign suggests
$a\,f(x)$	vertical dilation by a; negative a also reflects in the x-axis
$f(bx)$	horizontal dilation by $\frac{1}{b}$ – again the reciprocal, not b
the reason	the inside of the function acts on the input before the rule runs, so its effects are inverted

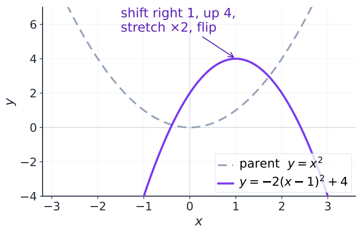
> Inside the brackets, everything is backwards. Say that once and the four rules stop needing to be memorised separately.

Combining shift, stretch and reflect

Combine additive shifts and multiplicative stretches to model a shifted, scaled version of a known shape.

> Read the general form $a\,f\big(b(x-h)\big)+k$ from the inside out.

Combining shift, stretch and reflect, in detail



> Combine additive shifts and multiplicative stretches to model a shifted, scaled version of a known shape.

5 Modelling

Choosing and building a model

Match the **model** 模型 to how the quantity changes:

- constant rate → linear
- constant second difference → quadratic
- several turns → a higher-degree polynomial

> State the **assumptions** 假设 – a model is only valid where they hold.

To construct one, use the given features (points, zeros with multiplicity, end behaviour), then check the answer against the domain that makes sense and interpret with real-world units.

Exam tips

- Read a function through its **rate of change**: average rate = secant slope; a constant rate means linear, a linear-changing rate means quadratic.
- Factor a polynomial to find **zeros** (x-intercepts) and their multiplicity; even multiplicity touches, odd crosses the axis.
- End behaviour is set by the **leading term**; a rational function's asymptotes come from the degrees of top and bottom.
- Build models from key features (points, zeros, end behaviour) and state your assumptions.
- Describe transformations precisely: shift $f(x-h)+k$, stretch $af(bx)$, reflect $-f(x)$ / $f(-x)$.

6 Recap

Unit 1 – key takeaways

1 Rates

avg rate = secant slope; constant → linear, linear → quadratic

2 Zeros

even multiplicity touches, odd crosses; n complex zeros

3 End behaviour

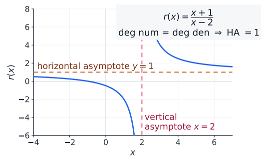
the leading term decides both ends

4 Rational

degrees give the asymptote; a cancelled factor is a hole

Check yourself

1. At a zero of even multiplicity, does the graph cross or touch?
2. Give the horizontal asymptote of $\frac{2x^2+3}{x^2-1}$.
3. Which term decides a polynomial's end behaviour?
4. A factor cancels in a rational function. Asymptote or hole?



Answers **1.** touches (tangent) **2.** $y = 2$ **3.** the leading term **4.** a hole