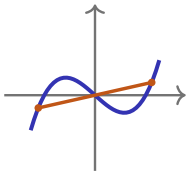


Polynomial & Rational Functions

AP Precalculus ■ Unit 1

AP Precalculus



What we will cover

- 1 Change in tandem
- 2 Rates of change
- 3 Polynomials: zeros & end behavior
- 4 Rational functions
- 5 Asymptotes & holes
- 6 Transformations
- 7 Modelling

1

Change in tandem

Functions, and change in tandem

A **function** 函数 maps each input to **exactly one** output. Inputs form the **domain** 定义域; outputs the **range** 值域.

■ **increasing** 递增:

$$a < b \Rightarrow f(a) < f(b)$$

■ **decreasing** 递减:

$$a < b \Rightarrow f(a) > f(b)$$



Bridge curve

The vocabulary every function question uses

a function 函数

a rule mapping each input to **exactly one** output

domain 定义域 **and**
range 值域

the allowed inputs, and the outputs they produce

independent 自变量 / **de-**
pendent 因变量

the input you choose, and the output it determines

local extremum 局部极值

a value larger (or smaller) than everything **nearby**

global maxi-
mum 全局最大值

larger than everything **in the whole domain** – a local one need not be global

piecewise-defined 分段函数

different rules over non-overlapping intervals; pick the branch whose interval contains your input

“Exactly one output” is the whole definition, and it is what the vertical-line test checks on a graph.

The words a function question uses

Independent and dependent

The **independent variable** is the input, plotted horizontally; the **dependent variable** is the output that follows.

Global (absolute) maximum

A **global (absolute) maximum** is the largest output anywhere in the domain; a **local (relative) extremum** is largest or smallest only nearby.

Piecewise-defined function

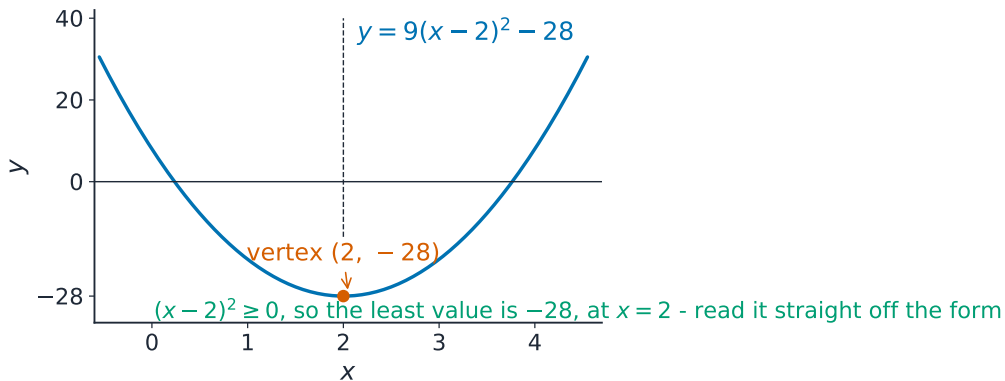
A **piecewise-defined function** uses a different rule on each interval – so check the endpoints, where the rules meet.

Increasing and decreasing

Say over *which interval*: a function is rarely increasing everywhere, and the interval is part of the answer.

Describe a graph in this order: domain and range, increase and decrease, extrema, end behaviour.

Completing the square gives the vertex of a parabola



Completing the square gives the vertex of a parabola

Average rate of change

The **average rate of change** 平均变化率 over $[a, b]$ is the **slope of the secant line** 割线:

$$\text{avg rate} = \frac{f(b) - f(a)}{b - a}.$$

Positive rate: both quantities move the **same** way. **Negative**: opposite ways.

$$f(x) = x^2$$

Over $[1, 4]$: $\frac{16 - 1}{3} = 5$.

Over $[1, 2]$: $\frac{4 - 1}{1} = 3$. The rate **itself changes** – which is exactly why f is not linear.

The rate of the rate

Linear 线性

average rate is **constant** – so
the rate of the rate is **zero**

Quadratic 二次

rates over equal intervals form
a **linear** pattern – so they
change at a **constant** rate



Parabola bridge

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Polynomials

Polynomials – degree, leading term, extrema

$$p(x) = a_n x^n + \cdots + a_1 x + a_0, \quad a_n \neq 0$$

Degree 次数 n ; **leading term** 首项 $a_n x^n$;
leading coefficient 首项系数 a_n .

- where it switches increasing/decreasing:
a **local extremum** 局部极值
- between two distinct real zeros there is
at least one extremum
- an even-degree polynomial has a global
max or a global min

A **point of inflection** 拐点
is where concavity changes –
concave up 上凹 to **concave
down** 下凹 or back.

That is where the rate of change switches
between increasing and decreasing.

Symmetry, degree and end behaviour

even 偶函数

$f(-x) = f(x)$ – symmetric across the y -axis, like x^2 and x^4

odd 奇函数

$f(-x) = -f(x)$ – symmetric about the **origin**, like x and x^3

successive differences 逐次差分

the degree is the least n for which the n th differences of **equally spaced** outputs become constant

end behavior 末端行为

where the output heads as $x \rightarrow \pm\infty$: set by the **leading term** alone

the two cases

even degree – both ends go the same way; odd degree – opposite ways. The leading coefficient's sign decides which

Every other term becomes irrelevant far from the origin. That is why end behaviour needs only the degree and the leading coefficient.

Zeros, multiplicity and complex roots

a is a **zero** 零点 when $p(a) = 0$; then $(x - a)$ is a factor. Repeated n times $\Rightarrow$ **multiplicity** 重数 n .

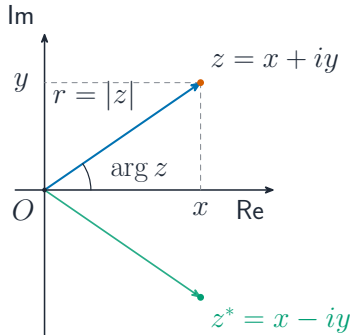
Counting multiplicities, a degree- n polynomial has exactly n **complex** 复数 zeros.

Non-real zeros come in **conjugate** 共轭 pairs:
 $a + bi$ and $a - bi$.

Even multiplicity
the graph is **tangent** – it touches, does not cross

Odd multiplicity
the graph **crosses** the axis

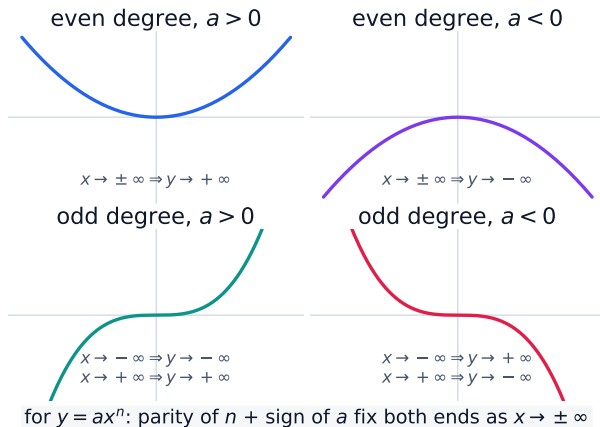
A complex number on an Argand diagram



A complex number on an Argand diagram: modulus is the distance, argument the angle

End behaviour is decided by the leading term

For large $|x|$, the leading term $a_n x^n$ **dominates** 主导 everything else – so the **sign of a_n** and the **parity of n** fix both ends:



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Rational functions

Rational end behaviour – compare the degrees

top $>$ bottom
follows the **quotient** polynomial;
if linear, a **slant asymptote** 斜渐近线

equal degrees
horizontal asymptote 水平渐近线
at the **ratio of leading coefficients**

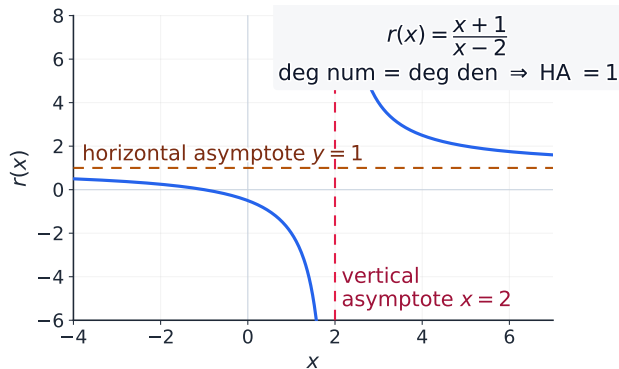
top $<$ bottom
the quotient
 $\rightarrow 0$, so $y = 0$

At a horizontal asymptote $y = b$, the outputs get **and stay** arbitrarily close to b .

Vertical asymptotes and holes

- a **vertical asymptote** 垂直渐近线 at $x = a$: a is a zero of the **denominator** that does **not** cancel
- a **hole** 空洞 at $x = c$: the factor **cancels** – a single missing point at (c, L)

Check **each side** of an asymptote – the two sides can go opposite ways.



Worked example – describe a rational function

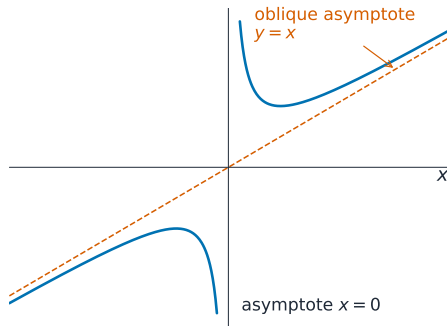
$$r(x) = \frac{2x^2 + 3}{x^2 - 1}$$

Equal degrees, so the horizontal asymptote is

$$y = \frac{2}{1} = 2.$$

$x^2 - 1 = 0$ at $x = \pm 1$, and neither cancels, so there are vertical asymptotes at

$$x = 1 \quad \text{and} \quad x = -1.$$



Top degree **one more** than the bottom: long division gives a **slant** asymptote.

Equivalent forms reveal different features

Factored form

shows real **zeros** – and hence intercepts, holes, vertical asymptotes, domain and range

Standard form

shows the **degree** and **leading term** – and hence end behaviour

Long division 多项式长除法 rewrites $f = gq + r$; the **quotient** 商 q is the **slant asymptote** when the degrees differ by one.

The same expression, written to reveal different things

factored formshows the **zeros** immediately**standard form**shows the **degree** and the leading coefficient, so the end behaviour**vertex form**shows the **vertex** of a quadratic without any calculation**after polynomial long division** 多项式长除法shows a rational function's **asymptote** as the quotient, plus a remainder that vanishes**the habit**before calculating, ask which form already *contains* the answer

Rewriting is not busywork; it is choosing the representation in which the question is already answered.

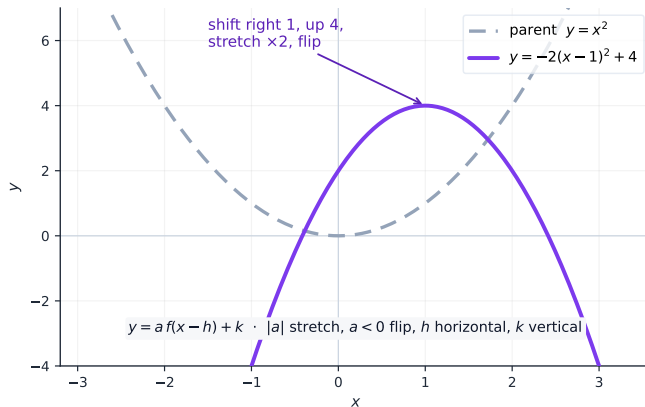
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Transformations

Transformations of functions

- **Translations** 平移: $f(x) + k$ up/down; $f(x - h)$ right/left
- **Dilations** 伸缩: $a f(x)$ vertical; $f(bx)$ horizontal
- **Reflections** 反射: $-f(x)$ over the x -axis; $f(-x)$ over the y -axis

Transformations of functions, in detail



Transforming the parent parabola: shift, stretch, and reflect

The four transformations, and the direction each moves

$f(x) + k$

vertical **shift** up by k – as expected

$f(x + k)$

horizontal shift **left** by k – the *opposite* of what the sign suggests

$a f(x)$

vertical **dilation** by a ; negative a also reflects in the x -axis

$f(bx)$

horizontal dilation by $\frac{1}{b}$ – again the reciprocal, not b

the reason

the inside of the function acts on the **input** before the rule runs, so its effects are inverted

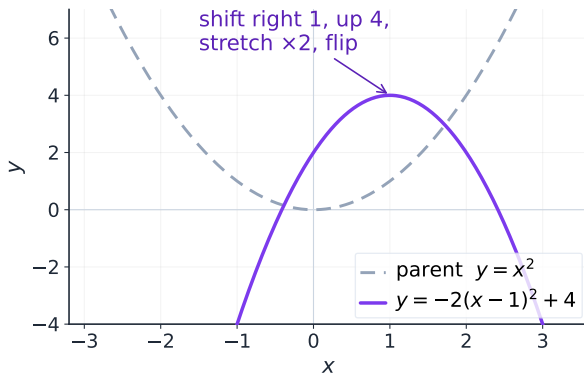
Inside the brackets, everything is backwards. Say that once and the four rules stop needing to be memorised separately.

Combining shift, stretch and reflect

Combine **additive** shifts and **multiplicative** stretches to model a shifted, scaled version of a known shape.

Read the general form $a f(b(x - h)) + k$ from the **inside out**.

Combining shift, stretch and reflect, in detail



Combine **additive** shifts and **multiplicative** stretches to model a shifted, scaled version of a known shape.

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Modelling

Choosing and building a model

Match the **model** 模型 to how the quantity changes:

- constant rate → linear
- constant second difference → quadratic
- several turns → a higher-degree polynomial

State the **assumptions** 假设 – a model is only valid where they hold.

To **construct** one, use the given features (points, zeros with multiplicity, end behaviour), then check the answer against the **domain that makes sense** and interpret with real-world units.

Exam tips

- Read a function through its **rate of change**: average rate = secant slope; a constant rate means linear, a linear-changing rate means quadratic.
- Factor a polynomial to find **zeros** (x-intercepts) and their multiplicity; even multiplicity touches, odd crosses the axis.
- End behaviour is set by the **leading term**; a rational function's asymptotes come from the degrees of top and bottom.
- Build models from key features (points, zeros, end behaviour) and state your assumptions.
- Describe transformations precisely: shift $f(x - h) + k$, stretch $af(bx)$, reflect $-f(x)$ / $f(-x)$.

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Recap

Unit 1 – key takeaways

1

Rates

avg rate = secant slope; constant
→ linear, linear → quadratic

2

Zeros

even multiplicity touches, odd
crosses; n complex zeros

3

End behaviour

the leading term decides both
ends

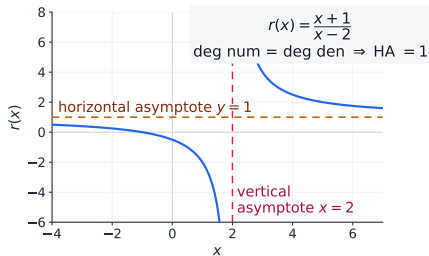
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Rational

degrees give the asymptote; a
cancelled factor is a hole

Check yourself

- 1 At a zero of even multiplicity, does the graph cross or touch?
- 2 Give the horizontal asymptote of $\frac{2x^2+3}{x^2-1}$.
- 3 Which term decides a polynomial's end behaviour?
- 4 A factor cancels in a rational function. Asymptote or hole?



Answers 1. touches (tangent) 2. $y = 2$ 3. the leading term 4. a hole