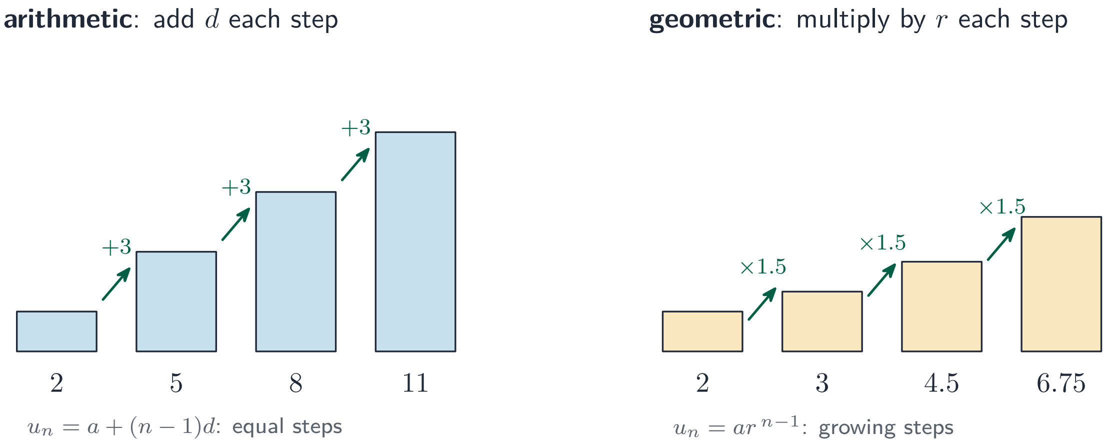
Exponential and Logarithmic Functions

AP Precalculus

Change in Arithmetic and Geometric Sequences

A **sequence** 数列 is a function from the whole numbers to the real numbers, so its graph is **discrete points**, not a curve.



An arithmetic sequence climbs in equal steps; a geometric one multiplies by a ratio

- An **arithmetic sequence** 等差数列 has a **common difference** 公差 d (a constant rate of change): $a_n = a_0 + dn$, or from a known term, $a_n = a_k + d(n-k)$.
- A **geometric sequence** 等比数列 has a **common ratio** 公比 r (a constant proportional change): $g_n = g_0\, r^{\,n}$, or $g_n = g_k\, r^{\,(n-k)}$.

An increasing arithmetic sequence grows by the **same amount** each step; an increasing geometric sequence grows by a **larger amount** each step.

Worked example. An arithmetic sequence with $a_0 = 3$ and $d = 5$ has $a_4 = 3+5(4) = 23$. A geometric sequence with $g_0 = 2$ and $r = 3$ has $g_4 = 2 \cdot 3^4 = 162$ – addition versus multiplication makes the geometric one pull far ahead.

Change in Linear and Exponential Functions

A sprinter: exponential and logarithmic models capture growth and decay rates in context

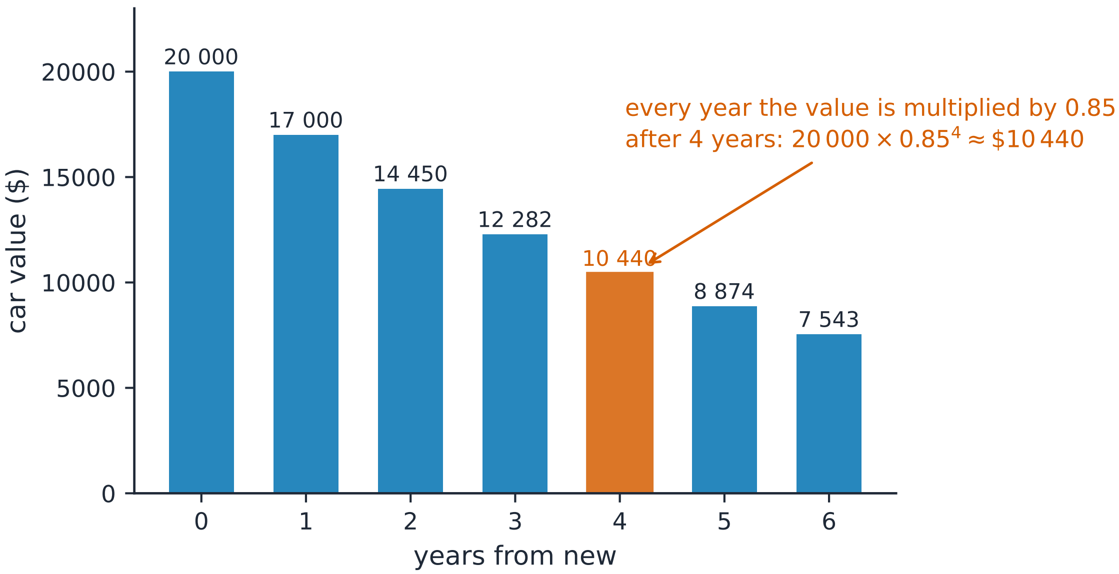
Sequences have continuous cousins:

- **Linear functions** $f(x) = b + mx$ mirror arithmetic sequences – an initial value plus repeated addition of the slope m. Point form: $f(x) = y_i + m(x - x_i)$.
- **Exponential functions** $f(x) = ab^x$ mirror geometric sequences – an initial value times repeated multiplication by the base b. Point form: $f(x) = y_i\, r^{(x-x_i)}$.

The difference: linear = repeated **addition**, exponential = repeated **multiplication**. (A sequence and its function may have different domains.)

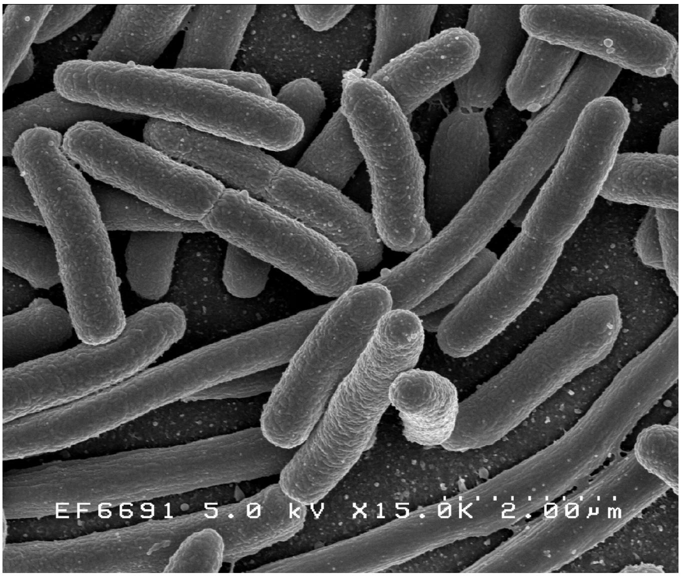
Exponential Functions

The general **exponential function** 指数函数 is $f(x) = ab^x$ with **initial value** 初始值 $a \neq 0$ and **base** 底数 $b > 0,\ b \neq 1$. Domain: all real numbers.



Exponential decay: a fixed percentage is lost each period

- $a > 0,\ b > 1$ gives **exponential growth** 指数增长; $a > 0,\ 0 < b < 1$ gives **exponential decay** 指数衰减.
- Outputs are **proportional** over equal-length input intervals. So an exponential is always increasing or always decreasing, and always concave up or always concave down – it has **no extrema** (except on a closed interval) and **no points of inflection**.



Bacteria under a microscope: exponential models describe populations that grow by a constant factor each time step

Exponential Function Manipulation

The exponent rules reshape exponential expressions and connect to graph transformations:

- **Product:** $b^m b^n = b^{m+n}$. A horizontal shift b^{x+k} equals a vertical stretch ab^x with $a = b^k$.
- **Power:** $(b^m)^n = b^{mn}$. A horizontal stretch b^{cx} equals a base change $(b^c)^x$.
- **Negative exponent:** $b^{-n} = \dfrac{1}{b^n}$.
- **Unit-fraction exponent:** $b^{1/k} = \sqrt[k]{b}$ (the kth **root** 方根).

Exponential Function Context and Data Modeling

Exponentials model quantities that grow by a **constant proportion** over equal intervals (repeated multiplication).

- Build a model from a ratio and an initial value, or from **two points** (solve the system for a and b).
- Sometimes a constant must be **added** to the data to reveal the proportional pattern.
- Use **exponential regression** 指数回归 on technology to fit a data set.
- The **natural base** 自然底数 $e \approx 2.718$ is the standard base for real-world models.

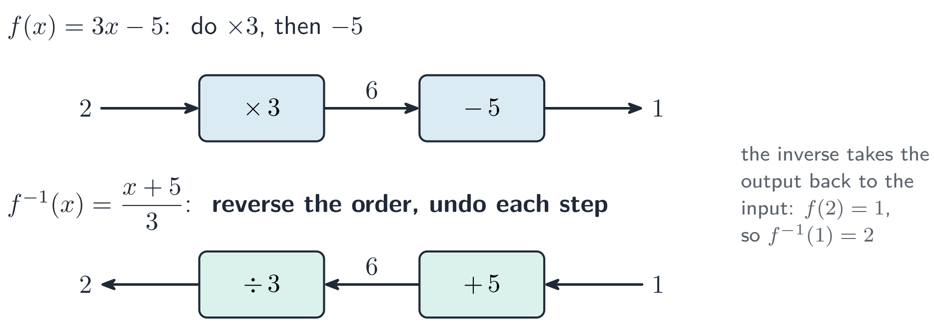
Competing Function Model Validation

When a rate of change shifts only slightly, linear, quadratic, and exponential models may all seem to fit. Choose using context and fit quality:

- A model is **appropriate** if its **residual plot** 残差图 shows **no pattern** (a **residual** 残差 is actual minus predicted).
- The **error** is the gap between predicted and actual; context decides whether an over- or under-estimate is safer.

Composition of Functions

The **composite function** 复合函数 $(f \circ g)(x) = f(g(x))$ feeds g's output into f. Its domain is the inputs of g whose outputs lie in f's domain.



A function as a machine; its inverse runs the machine backwards

- Composition is **not commutative**: $f(g(x))$ and $g(f(x))$ usually differ.
- To build $f(g(x))$ analytically, **substitute** $g(x)$ for every x in f.
- The **identity function** 恒等函数 $f(x) = x$ leaves any function unchanged under composition.
- A function can also be **decomposed** into simpler pieces – useful for seeing an additive shift as composing with $x + k$, or a dilation as composing with kx.

Matryoshka dolls: function composition nests layers — evaluate the inside first

Inverse Functions

A function is **invertible** 可逆 on a domain where each output comes from a **unique** input (you may restrict the domain to force this). The **inverse function** 反函数 f^{-1} reverses the mapping: if $f(a) = b$ then $f^{-1}(b) = a$.

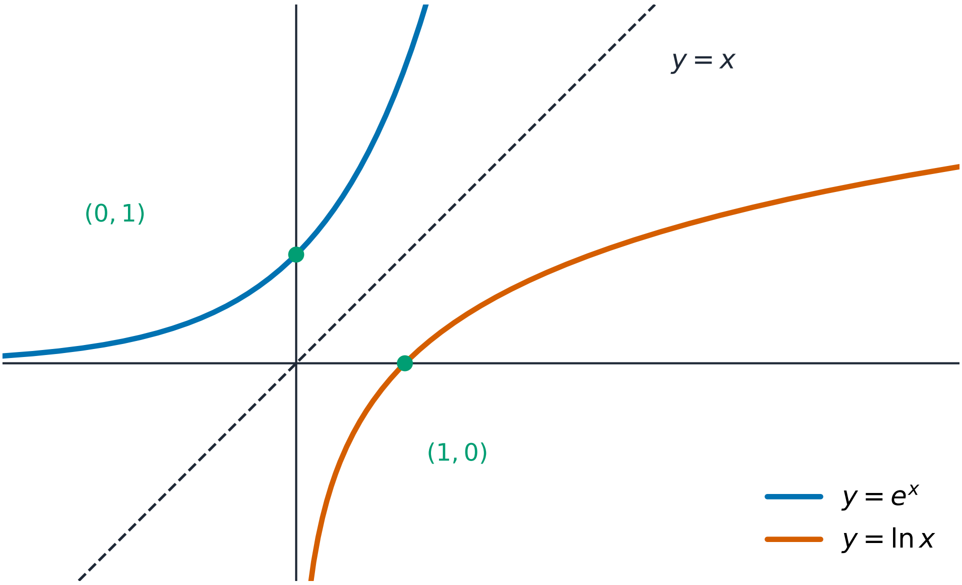
- $f\big(f^{-1}(x)\big) = f^{-1}\big(f(x)\big) = x$ (composition gives the identity).
- Domain and range **swap**: reverse each (a, b) to (b, a) in a table; reflect the graph over the line $y = x$.
- To find a formula: swap x and y in $y = f(x)$, then solve for y. Context may further limit where the inverse applies.

Logarithmic Expressions

The **logarithm** 对数 answers ”what exponent?”: $\log_b c = a$ means exactly $b^a = c$ (with $b > 0,\ b \neq 1$). An unwritten base means the **common logarithm** 常用对数 (base 10). On a **logarithmic scale**, each unit is a **multiplicative** step of the base (…, $10^0, 10^1, 10^2$, …).

Inverses of Exponential Functions

The logarithm is the **inverse of the exponential**: $y = b^x$ and $y = \log_b x$ undo each other, so their graphs are reflections over $y = x$. Hence $\log_b(b^x) = x$ and $b^{\log_b x} = x$. The **natural logarithm** 自然对数 $\ln x = \log_e x$ is the inverse of e^x.



e^x and ln x are reflections of each other in the line $y = x$

Logarithmic Functions

The **logarithmic function** 对数函数 $f(x) = \log_b x$ has domain $x > 0$ and range all reals. It is always increasing (for $b > 1$) or always decreasing (for $0 < b < 1$), always concave one way, with a **vertical asymptote** at $x = 0$ – the mirror image of the exponential's horizontal asymptote. It grows very slowly for large x.

Logarithmic Function Manipulation

The log properties reverse the exponent rules:

$$\log_b(xy) = \log_b x + \log_b y, \quad \log_b \frac{x}{y} = \log_b x - \log_b y, \quad \log_b(x^n) = n \log_b x.$$

The **change-of-base** 换底 formula lets you compute any log with technology: $\log_b x = \dfrac{\log x}{\log b} = \dfrac{\ln x}{\ln b}$.

Exponential and Logarithmic Equations and Inequalities

To solve, use the fact that exponentials and logs are inverses:

- Isolate the exponential, then take a log of both sides (bring the exponent down with the power property).
- Isolate the log, then exponentiate both sides.
- Always check for **extraneous solutions** – the argument of a log must stay **positive**.

Worked example. Solve $2 \cdot 3^x = 54$. Divide by 2: $3^x = 27 = 3^3$, so $x = 3$. When the sides are not tidy powers, take logs instead: $5^x = 20$ gives $x = \dfrac{\ln 20}{\ln 5} \approx 1.86$.

Logarithmic Function Context and Data Modeling

Logarithms model quantities that change over huge multiplicative ranges (sound, acid strength, earthquakes). Build a log model from data, and use **logarithmic regression** for a data set. Because a log compresses large values, it turns proportional growth into a straight-line pattern – the idea behind semi-log plots.

Semi-log Plots

A **semi-log plot** 半对数图 puts the output on a **logarithmic** axis and the input on a normal axis. On these axes, an **exponential** function $y = ab^x$ becomes a **straight line**, because $\log y = \log a + (\log b)\,x$ is linear in x. So: if data look linear on a semi-log plot, an exponential model fits; the line's slope gives $\log b$ and its intercept gives $\log a$.

Exam tips

- Distinguish **arithmetic** (add a common difference) from **geometric** (multiply a common ratio) sequences and their linear/exponential function cousins.
- Exponential = repeated **multiplication**, so it eventually outgrows any linear or polynomial model.
- The **logarithm is the inverse** of the exponential ($\log_b c = a \Leftrightarrow b^a = c$); use it to solve $b^x = k$.
- Apply the log laws (product, quotient, power) and the change-of-base formula; the argument of a log must be **positive**.
- On a **semi-log plot** an exponential model becomes a straight line.