Polynomial and Rational Functions

AP Precalculus

Change in Tandem

A **function** 函数 is a rule that maps each input to **exactly one** output. The set of allowed inputs is the **domain** 定义域; the set of outputs is the **range** 值域. The input variable is the **independent variable** 自变量 and the output variable is the **dependent variable** 因变量. A function rule can be shown graphically, numerically, analytically (a formula), or verbally.

As the input changes, the output changes "in tandem" – together. Over an interval, a function is:

- **increasing** 递增 if, whenever $a < b$, then $f(a) < f(b)$ (bigger input, bigger output);
- **decreasing** 递减 if, whenever $a < b$, then $f(a) > f(b)$ (bigger input, smaller output).

A graph of a function shows all its input–output pairs, so you can read this behavior straight off the picture.

Rates of Change

The **average rate of change** 平均变化率 of a function over an interval is the change in output divided by the change in input – the single constant rate that would give the same total change. Over $[a, b]$:

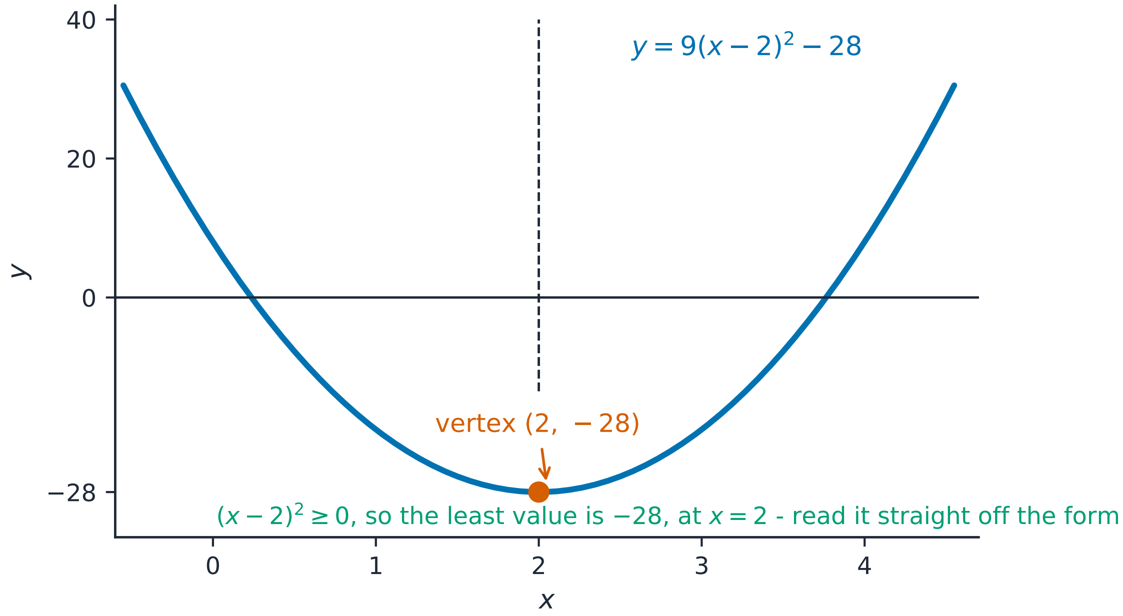
$$\text{avg rate} = \frac{f(b) - f(a)}{b - a}.$$

The **rate of change at a point** measures how fast the output changes right at that input. You **approximate** it with average rates over small intervals around the point. Comparing two points, the one with the larger average rate (over small enough intervals) is changing faster. A **positive** rate means both quantities move the same way; a **negative** rate means they move in opposite ways.

Worked example. For $f(x) = x^2$, the average rate of change over $[1, 4]$ is $\dfrac{f(4) - f(1)}{4 - 1} = \dfrac{16 - 1}{3} = 5$. Over $[1, 2]$ it is $\dfrac{4 - 1}{1} = 3$ – the rate itself changes, which is exactly why f is not linear.

Rates of Change in Linear and Quadratic Functions

The average rate of change over $[a, b]$ is the slope of the **secant line** 割线 from $(a, f(a))$ to $(b, f(b))$.



Completing the square gives the vertex of a parabola

- For a **linear** 线性 function, the average rate of change over any interval is **constant** – so the rate at which the rate changes is zero.
- For a **quadratic** 二次 function, the average rates of change over equal-length intervals themselves form a **linear** pattern – so those rates change at a **constant** rate.

This "rate of the rate" idea distinguishes function types: a constant second difference signals a quadratic.

A hanging bridge cable takes the shape of a parabola-like curve — a quadratic you can see in steel

Polynomial Functions and Rates of Change

A smooth bridge curve: polynomial and rational graphs model real shapes with asymptotes and turns

A nonconstant **polynomial** 多项式 has the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0.$$

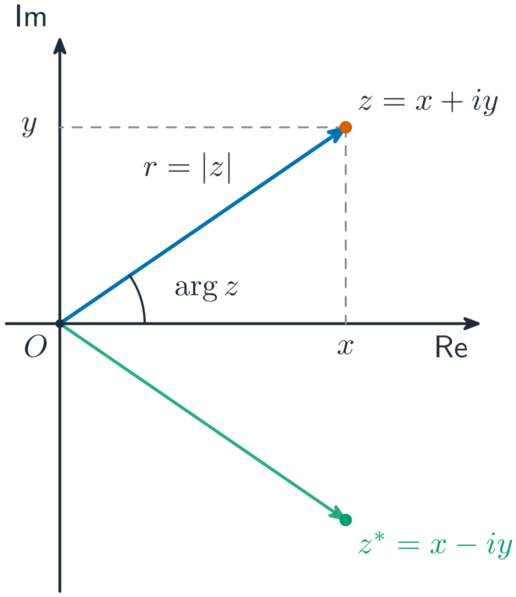
Its **degree** 次数 is n, its **leading term** 首项 is $a_n x^n$, and its **leading coefficient** 首项系数 is a_n. A constant is a polynomial of degree 0.

Key features:

- Where a polynomial switches between increasing and decreasing, it has a **local (relative) extremum** 局部极值. The greatest local maximum is the **global (absolute) maximum** 全局最大值 *only when the polynomial has one* — but most polynomials are unbounded (every odd-degree polynomial runs off to $\pm\infty$), so they have no global maximum or minimum at all.
- Between any two distinct real zeros there is at least one local maximum or minimum.
- An **even-degree** polynomial has either a global maximum or a global minimum.
- A **point of inflection** 拐点 is where the graph changes concavity – from **concave up** 上凹 to **concave down** 下凹 or the reverse – i.e. where the rate of change switches between increasing and decreasing.
- A function has **symmetry** when it is **even** 偶函数 – $f(-x) = f(x)$, graph symmetric across the y-axis (like x^2, x^4) – or **odd** 奇函数 – $f(-x) = -f(x)$, graph symmetric by rotation about the origin (like x^3, x^5). Test by substituting $-x$ and simplifying; later, $\cos\theta$ is even and $\sin\theta$ is odd.

Polynomial Functions and Complex Zeros

A **zero** 零点 (or **root** 根) of p is a number a with $p(a) = 0$. For a real a, $(x - a)$ is a factor of p exactly when a is a zero. If the factor $(x - a)$ is repeated n times, that zero has **multiplicity** 重数 n. Counting multiplicities, a degree-n polynomial has exactly n **complex** 复数 zeros.

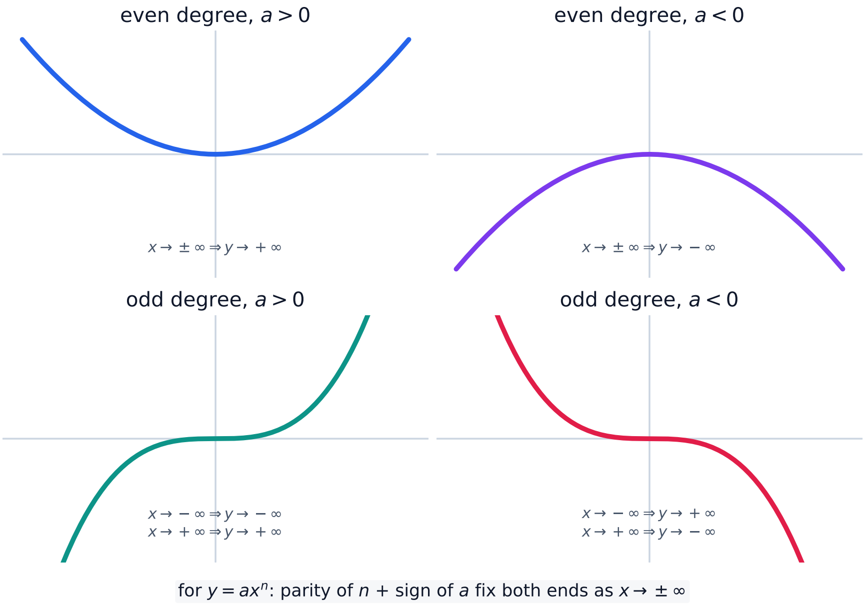


A complex number on an Argand diagram: modulus is the distance, argument the angle

- A real zero a gives an x**-intercept** at $(a, 0)$; real zeros are the endpoints of the intervals where $p(x) \geq 0$ or ≤ 0.
- Non-real zeros come in **conjugate** 共轭 pairs: if $a + bi$ is a zero, so is $a - bi$.
- At a real zero of **even** multiplicity the graph is **tangent** to the x-axis (it touches but does not cross); at odd multiplicity it crosses.
- The degree equals the least n for which the nth **successive differences** of equally-spaced outputs become constant.

Polynomial Functions and End Behavior

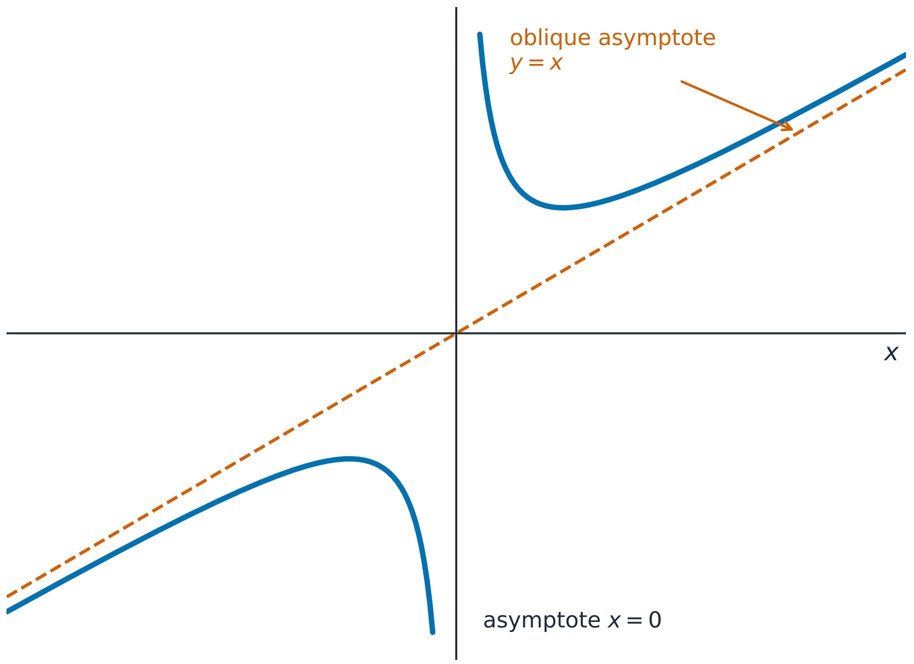
End behavior 末端行为 describes where a function heads as the input grows without bound. For a nonconstant polynomial, as $x \to \pm\infty$ the output goes to $+\infty$ or $-\infty$, written e.g. $\lim_{x\to\infty} p(x) = \infty$. Which way depends entirely on the **leading term** $a_n x^n$, because for large $|x|$ it **dominates** all lower-degree terms: the sign of a_n and whether n is even or odd fix both ends.



Every polynomial's ends are decided by its degree's parity and the sign of its leading coefficient

Rational Functions and End Behavior

A **rational function** 有理函数 is a quotient of two polynomials, $r(x) = \dfrac{\text{numerator}}{\text{denominator}}$. Its end behavior is governed by the **quotient of the leading terms**:



When the numerator has higher degree, the curve approaches a slanting asymptote

- **Numerator degree > denominator degree:** the quotient is a nonconstant polynomial, and r follows that polynomial's end behavior. If that quotient is linear, the graph has a **slant asymptote** 斜渐近线.
- **Equal degrees:** the quotient is a constant, giving a **horizontal asymptote** 水平渐近线 $y =$ the ratio of leading coefficients.
- **Numerator degree < denominator degree:** the quotient tends to 0, so the horizontal asymptote is $y = 0$.

At a horizontal asymptote $y = b$, the outputs get and stay arbitrarily close to b: $\lim_{x\to\pm\infty} r(x) = b$.

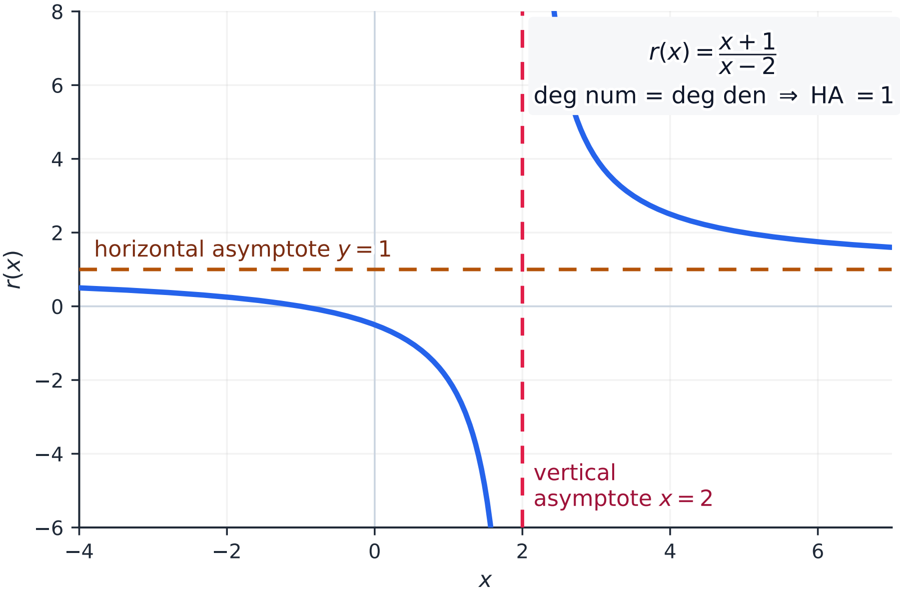
Rational Functions and Zeros

The real **zeros** of a rational function are the real zeros of its **numerator** that are still in the domain. These zeros, together with the zeros of the denominator, split the number line into the intervals you test when solving inequalities $r(x) \geq 0$ or $r(x) \leq 0$.

Rational Functions and Vertical Asymptotes

A **vertical asymptote** 垂直渐近线 occurs at $x = a$ when a is a zero of the **denominator** but not cancelled by the numerator – more precisely, when its multiplicity in the denominator exceeds its multiplicity in the numerator. Near it, the denominator is nearly zero, so r shoots to $\pm\infty$: $\lim_{x\to a^\pm} r(x) = \pm\infty$. Check each side, since the two sides can go opposite ways.

Worked example. Describe $r(x) = \dfrac{2x^2+3}{x^2-1}$. The numerator and denominator have **equal degree**, so the horizontal asymptote is $y = \dfrac{2}{1} = 2$. The denominator $x^2 - 1$ is zero at $x = \pm 1$ and neither cancels, so there are **vertical asymptotes** at $x = 1$ and $x = -1$.



A rational function approaching a vertical asymptote and a horizontal asymptote

Rational Functions and Holes

A **hole** 空洞 (removable point) occurs at $x = c$ when a factor cancels – the multiplicity of the zero c in the numerator is at least its multiplicity in the denominator. The graph is missing a single point. Its height is the limit of the simplified function: if inputs near c give outputs near L, the hole is at (c, L), and $\lim_{x\to c} r(x) = L$.

Equivalent Representations of Polynomial and Rational Expressions

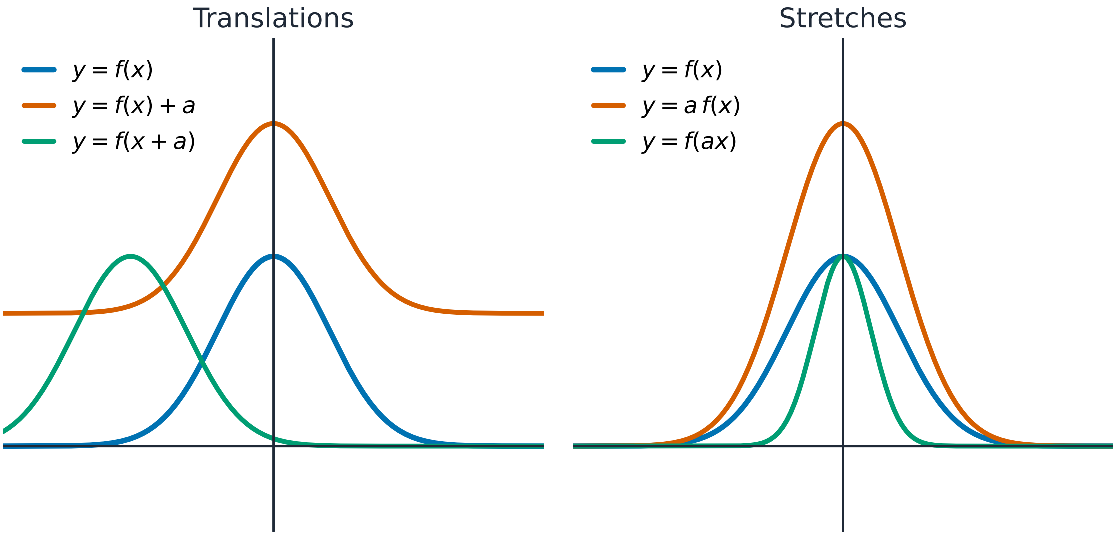
The same expression, written differently, reveals different features:

- **Factored form** shows real zeros, and hence x-intercepts, holes, vertical asymptotes, and domain.
- **Standard form** (expanded) shows the degree and leading term, and hence end behavior.

Polynomial long division 多项式长除法 rewrites $f(x) = g(x)\,q(x) + r(x)$, where q is the **quotient 商** and r the **remainder 余数** (of smaller degree than g). The quotient gives the equation of a **slant asymptote** when the numerator's degree is one more than the denominator's.

Transformations of Functions

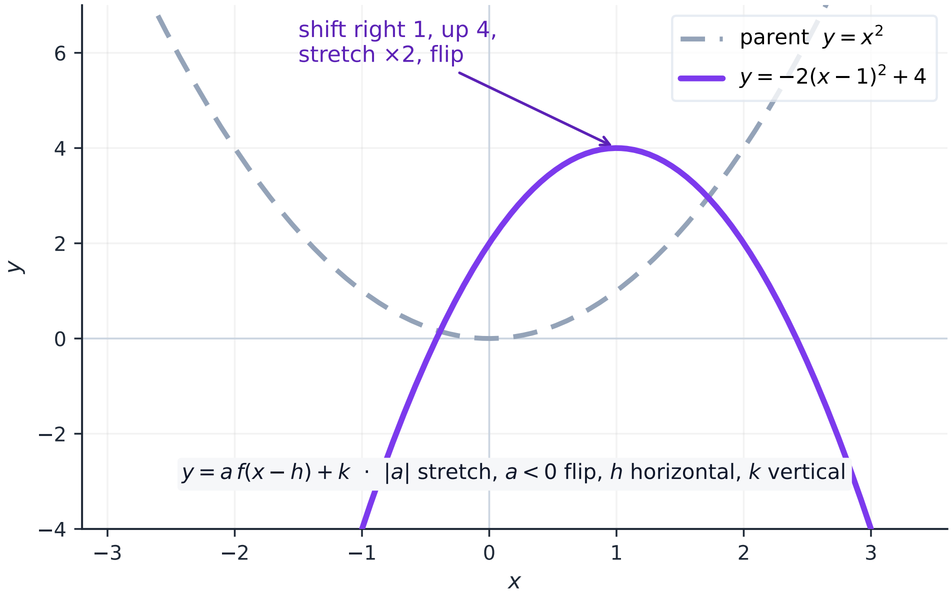
A **transformation 变换** builds a new function from an old one f:



Adding to the output or input shifts the curve; a multiplier stretches it

- **Translations 平移** (shifts): $f(x) + k$ moves up/down; $f(x - h)$ moves right/left.
- **Dilations 伸缩** (stretches): $a\,f(x)$ stretches vertically; $f(bx)$ stretches horizontally.
- **Reflections 反射**: $-f(x)$ flips over the x-axis; $f(-x)$ flips over the y-axis.

Combine additive shifts and multiplicative stretches to model shifted, scaled versions of a known shape.



Transforming the parent parabola: shift, stretch, and reflect

Function Model Selection and Assumption Articulation

Choosing a **model** 模型 means matching a function type to how a quantity changes. A linear model fits a constant rate of change; a quadratic fits a constant second difference; a polynomial fits data with several turns. In geometric situations the number of dimensions is a strong hint: quantities about **area** (two dimensions) are typically **quadratic**, and quantities about **volume** (three dimensions) are typically **cubic** – for example a box's volume as a function of a cut length. State the **assumptions** 假设 your model relies on (for example, that the pattern continues), because a model is only valid where those assumptions hold.

Function Model Construction and Application

To **construct** a model, use given features – points, intercepts, zeros with multiplicities, end behavior – to write the function, then use it to answer questions in context. Always check the answer against the **domain** that makes sense for the situation, and interpret outputs with their real-world units.

A **piecewise-defined function** 分段函数 uses different rules over non-overlapping intervals of the domain. To evaluate it, pick the branch whose interval contains the input. For example,

$$f(x) = \begin{cases} x^2, & x < 0 \\ 2x, & x \geq 0 \end{cases}$$

gives $f(-3) = 9$ from the first branch but $f(3) = 6$ from the second – useful when behaviour changes at a threshold (a tiered price, a speed limit that changes).

Exam tips

- Read a function through its **rate of change**: average rate = secant slope; a constant rate means linear, a linear-changing rate means quadratic.
- Factor a polynomial to find **zeros** (x-intercepts) and their multiplicity; even multiplicity touches, odd crosses the axis.
- End behaviour is set by the **leading term**; a rational function's asymptotes come from the degrees of top and bottom.
- Build models from key features (points, zeros, end behaviour) and state your assumptions.
- Describe transformations precisely: shift $f(x-h)+k$, stretch $af(bx)$, reflect $-f(x)$ / $f(-x)$.