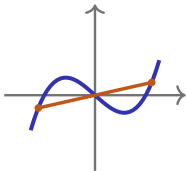


Polynomial & Rational Functions

AP Precalculus ■ Unit 1

AP Precalculus



What we will cover

- 1 Change in tandem
- 2 Rates of change
- 3 Polynomials: zeros & end behavior
- 4 Rational functions
- 5 Asymptotes & holes
- 6 Transformations
- 7 Modelling

1

Change in tandem

Functions, and change in tandem

A **function** maps each input to **exactly one** output. Inputs form the **domain**; outputs the **range**.

- **increasing** :
 $a < b \Rightarrow f(a) < f(b)$
- **decreasing** :
 $a < b \Rightarrow f(a) > f(b)$



Bridge curve

The vocabulary every function question uses

a function

a rule mapping each input to **exactly one** output

domain **and**
range

the allowed inputs, and the outputs they produce

independent / **de-**
pendent

the input you choose, and the output it determines

local extremum

a value larger (or smaller) than everything **nearby**

global maxi-
num

larger than everything **in the whole domain** – a local one need not be global

piecewise-defined

different rules over non-overlapping intervals; pick the branch whose interval contains your input

“Exactly one output” is the whole definition, and it is what the vertical-line test checks on a graph.

The words a function question uses

Independent and dependent

The **independent variable** is the input, plotted horizontally; the **dependent variable** is the output that follows.

Global (absolute) maximum

A **global (absolute) maximum** is the largest output anywhere in the domain; a **local (relative) extremum** is largest or smallest only nearby.

Piecewise-defined function

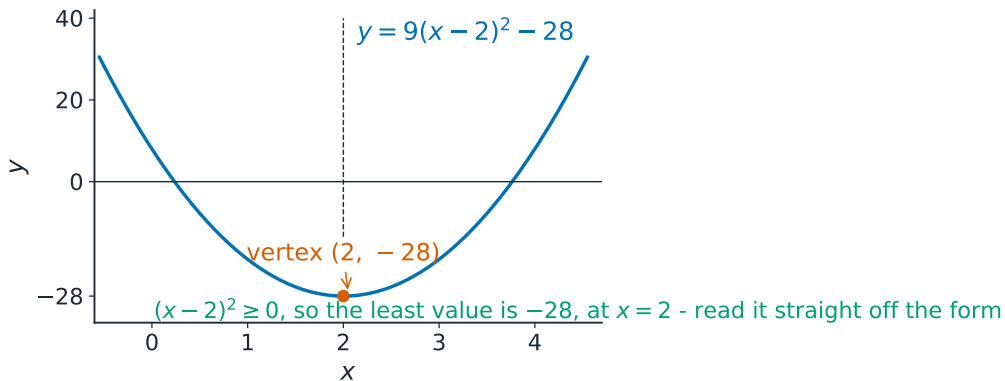
A **piecewise-defined function** uses a different rule on each interval – so check the endpoints, where the rules meet.

Increasing and decreasing

Say over *which interval*: a function is rarely increasing everywhere, and the interval is part of the answer.

Describe a graph in this order: domain and range, increase and decrease, extrema, end behaviour.

Completing the square gives the vertex of a parabola



Completing the square gives the vertex of a parabola

Average rate of change

The **average rate of change**

over $[a, b]$ is the **slope of the secant line**

:

$$\text{avg rate} = \frac{f(b) - f(a)}{b - a}.$$

Positive rate: both quantities move the same way. **Negative**: opposite ways.

$$f(x) = x^2$$

Over $[1, 4]$: $\frac{16 - 1}{3} = 5$.

Over $[1, 2]$: $\frac{4 - 1}{1} = 3$. The rate **itself changes** – which is exactly why f is not linear.

The rate of the rate

Linear

average rate is **constant** – so
the rate of the rate is **zero**

Quadratic

rates over equal intervals form
a **linear** pattern – so they
change at a **constant** rate



Parabola bridge

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Polynomials

Polynomials – degree, leading term, extrema

$$p(x) = a_n x^n + \cdots + a_1 x + a_0, \quad a_n \neq 0$$

Degree n ; **leading term** $a_n x^n$;
leading coefficient a_n .

- where it switches increasing/decreasing:
a **local extremum**
- between two distinct real zeros there is
at least one extremum
- an even-degree polynomial has a global
max or a global min

A point of inflection

is where concavity changes –
concave up to **concave down**
or back.

That is where the rate of change switches
between increasing and decreasing.

Symmetry, degree and end behaviour

even

$f(-x) = f(x)$ – symmetric across the y -axis, like x^2 and x^4

odd

$f(-x) = -f(x)$ – symmetric about the **origin**, like x and x^3

successive differences

the degree is the least n for which the n th differences of **equally spaced** outputs become constant

end behavior

where the output heads as $x \rightarrow \pm\infty$: set by the **leading term** alone

the two cases

even degree – both ends go the same way; odd degree – opposite ways. The leading coefficient's sign decides which

Every other term becomes irrelevant far from the origin. That is why end behaviour needs only the degree and the leading coefficient.

Zeros, multiplicity and complex roots

a is a **zero** when $p(a) = 0$; then $(x - a)$ is a factor. Repeated n times $\Rightarrow$ **multiplicity** n .

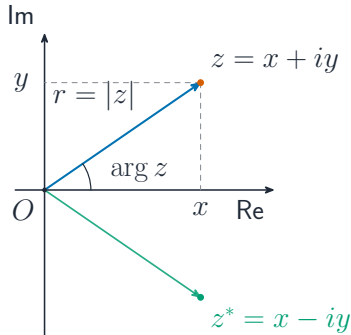
Counting multiplicities, a degree- n polynomial has exactly n **complex** zeros.

Non-real zeros come in **conjugate** pairs:
 $a + bi$ and $a - bi$.

Even multiplicity
the graph is **tangent** – it touches, does not cross

Odd multiplicity
the graph **crosses** the axis

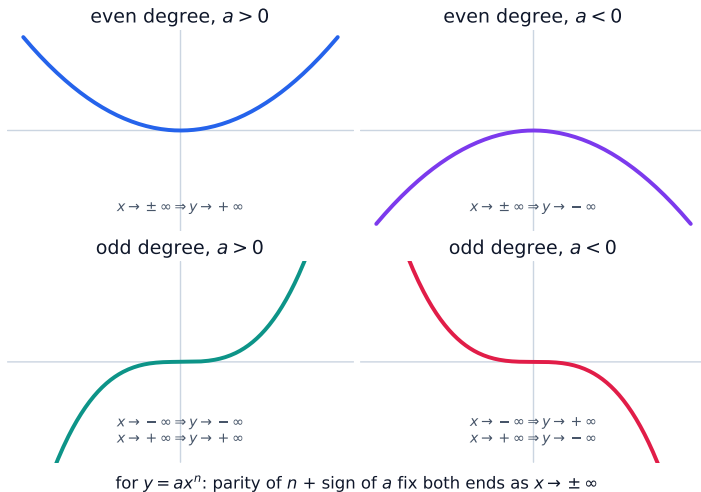
A complex number on an Argand diagram



A complex number on an Argand diagram: modulus is the distance, argument the angle

End behaviour is decided by the leading term

For large $|x|$, the leading term $a_n x^n$ **dominates** everything else – so the **sign of a_n** and the **parity of n** fix both ends:



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Rational functions

Rational end behaviour – compare the degrees

top $>$ bottom
follows the **quotient** polynomial;
if linear, a **slant asymptote**

equal degrees
horizontal asymptote
at the **ratio of leading coefficients**

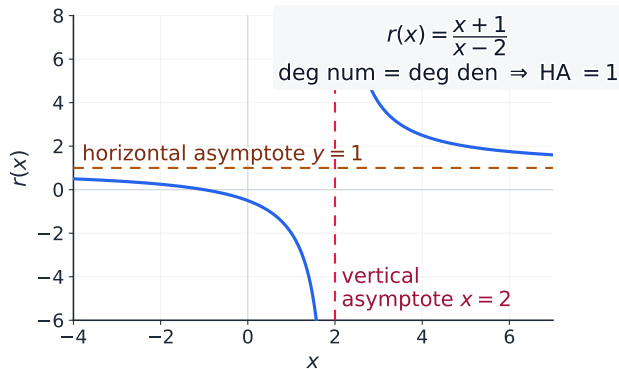
top $<$ bottom
the quotient
 $\rightarrow 0$, so $y = 0$

At a horizontal asymptote $y = b$, the outputs get **and stay** arbitrarily close to b .

Vertical asymptotes and holes

- a **vertical asymptote** at $x = a$: a is a zero of the **denominator** that does **not** cancel
- a **hole** at $x = c$: the factor **cancels** – a single missing point at (c, L)

Check **each side** of an asymptote – the two sides can go opposite ways.



Worked example – describe a rational function

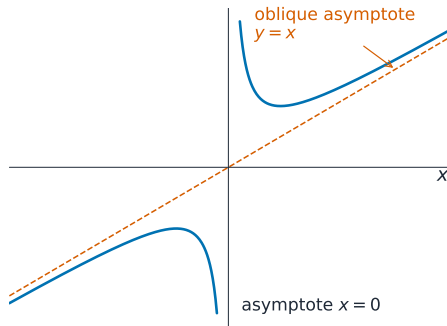
$$r(x) = \frac{2x^2 + 3}{x^2 - 1}$$

Equal degrees, so the horizontal asymptote is

$$y = \frac{2}{1} = 2.$$

$x^2 - 1 = 0$ at $x = \pm 1$, and neither cancels, so there are vertical asymptotes at

$$x = 1 \quad \text{and} \quad x = -1.$$



Top degree **one more** than the bottom: long division gives a **slant** asymptote.

Equivalent forms reveal different features

Factored form

shows real **zeros** – and hence intercepts, holes, vertical asymptotes, domain and range

Standard form

shows the **degree** and **leading term** – and hence end behaviour

Long division

rewrites $f = gq + r$; the **quotient** q is the **slant asymptote** when the degrees differ by one.

The same expression, written to reveal different things

factored formshows the **zeros** immediately**standard form**shows the **degree** and the leading coefficient, so the end behaviour**vertex form**shows the **vertex** of a quadratic without any calculation**after polynomial long division**shows a rational function's **asymptote** as the quotient, plus a remainder that vanishes**the habit**before calculating, ask which form already *contains* the answer

Rewriting is not busywork; it is choosing the representation in which the question is already answered.

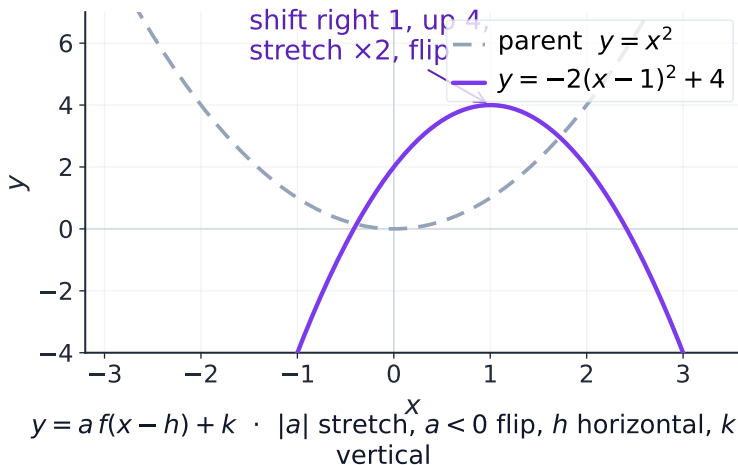
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Transformations

Transformations of functions

- **Translations** : $f(x) + k$ up/down; $f(x - h)$ right/left
- **Dilations** : $a f(x)$ vertical; $f(bx)$ horizontal
- **Reflections** : $-f(x)$ over the x -axis; $f(-x)$ over the y -axis

Transformations of functions, in detail



Transforming the parent parabola: shift, stretch, and reflect

The four transformations, and the direction each moves

$f(x) + k$

vertical **shift** up by k – as expected

$f(x + k)$

horizontal shift **left** by k – the *opposite* of what the sign suggests

$a f(x)$

vertical **dilation** by a ; negative a also reflects in the x -axis

$f(bx)$

horizontal dilation by $\frac{1}{b}$ – again the reciprocal, not b

the reason

the inside of the function acts on the **input** before the rule runs, so its effects are inverted

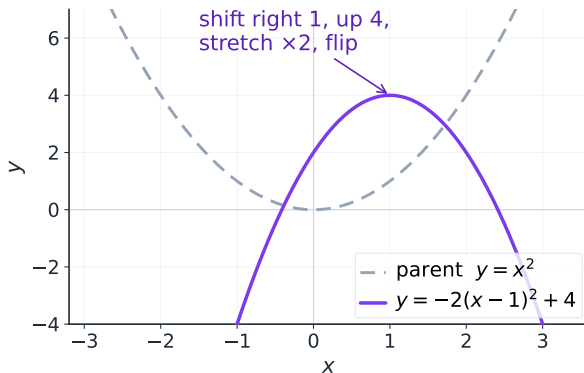
Inside the brackets, everything is backwards. Say that once and the four rules stop needing to be memorised separately.

Combining shift, stretch and reflect

Combine **additive** shifts and **multiplicative** stretches to model a shifted, scaled version of a known shape.

Read the general form $a f(b(x - h)) + k$ from the **inside out**.

Combining shift, stretch and reflect, in detail



Combine **additive** shifts and **multiplicative** stretches to model a shifted, scaled version of a known shape.

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Modelling

Choosing and building a model

Match the **model** to **how the quantity changes**:

- constant rate → **linear**
- constant second difference → **quadratic**
- several turns → a higher-degree **polynomial**

State the **assumptions** –
a model is only valid where they
hold.

To **construct** one, use the given features (points, zeros with multiplicity, end behaviour), then check the answer against the **domain that makes sense** and interpret with real-world units.

Exam tips

- Read a function through its **rate of change**: average rate = secant slope; a constant rate means linear, a linear-changing rate means quadratic.
- Factor a polynomial to find **zeros** (x-intercepts) and their multiplicity; even multiplicity touches, odd crosses the axis.
- End behaviour is set by the **leading term**; a rational function's asymptotes come from the degrees of top and bottom.
- Build models from key features (points, zeros, end behaviour) and state your assumptions.
- Describe transformations precisely: shift $f(x - h) + k$, stretch $af(bx)$, reflect $-f(x)$ / $f(-x)$.

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Recap

Unit 1 – key takeaways

1

Rates

avg rate = secant slope; constant
→ linear, linear → quadratic

2

Zeros

even multiplicity touches, odd
crosses; n complex zeros

3

End behaviour

the leading term decides both
ends

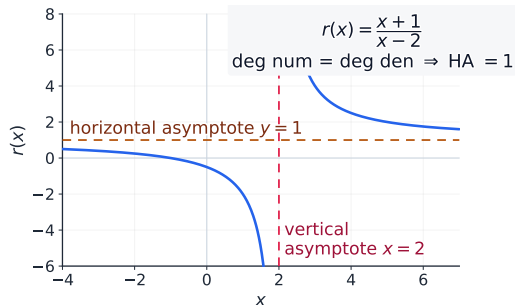
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Rational

degrees give the asymptote; a
cancelled factor is a hole

Check yourself

- 1 At a zero of even multiplicity, does the graph cross or touch?
- 2 Give the horizontal asymptote of $\frac{2x^2+3}{x^2-1}$.
- 3 Which term decides a polynomial's end behaviour?
- 4 A factor cancels in a rational function. Asymptote or hole?



Answers 1. touches (tangent) 2. $y = 2$ 3. the leading term 4. a hole