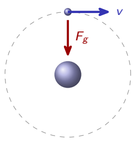
# Energy & Momentum of Rotating Systems

AP Physics C: Mechanics ▪ Unit 6

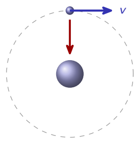
AP Physics C: Mechanics






## What we will cover






# 1 Rotational KE



## Rotational kinetic energy

**Energy of spin**

$$K_{rot} = \tfrac{1}{2}I\omega^2$$

A rolling body has both: $K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$.

A **flywheel** 飞轮 stores energy in its spin – and $K \propto \omega^2$.

**Torque and work**

$$W = \int \tau\, d\theta, \qquad P = \tau\omega$$

Area under a $\tau$–$\theta$ graph; $W_{net} = \Delta K_{rot}$.

$\tau = 15$, $\Delta\theta = 8$: $W = 120\,\mathrm{J}$; at $\omega = 20$, $P = 300\,\mathrm{W}$.

## The whole energy toolkit, translated

| | |
|---|---|
| **kinetic energy** | $\frac{1}{2}mv^2 \longrightarrow \frac{1}{2}I\omega^2$ |
| **work** | $W = \int F\,dx \longrightarrow W = \int \tau\, d\theta$ |
| **power** | $P = Fv \longrightarrow P = \tau\omega$ |
| **momentum** | $\vec{p} = m\vec{v} \longrightarrow \vec{L} = I\vec{\omega}$ |
| **the substitution** | $F \to \tau, \quad x \to \theta, \quad v \to \omega, \quad m \to I$ |

There is nothing new to learn in rotational energy – only a dictionary. Every translational result you already have comes across intact.

# 2 Angular momentum

## Angular momentum and impulse

**Angular momentum** 角动量 is the rotational twin of momentum:

**Point particle and rigid body**

$$\vec{L} = \vec{r} \times \vec{p}, \qquad L = I\omega$$

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$$

**Worked example**

$L_i = 30$; $\tau = 5\,\mathrm{N\,m}$ for $4\,\mathrm{s}$:

$$\Delta L = 5 \times 4 = 20$$

$$L_f = 30 + 20 = 50\,\mathrm{kg\,m^2/s}$$

**Angular impulse** 角冲量 $\int \tau\, dt = \Delta L$.



## Conservation of angular momentum

With zero net external torque, angular momentum is conserved – even as the shape changes:

**Before = after**

$$I_i \omega_i = I_f \omega_f$$

A skater pulls their arms in: $I$ drops, so $\omega$ rises. (Kinetic energy rises too – paid by muscular work.)

*Skater spin*



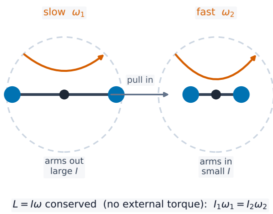
## Worked example – a bullet strikes a pivoted rod

**A 0.020 kg bullet at 300 m/s hits the tip of a uniform rod ($M = 1.5$ kg, $l = 0.60$ m) pivoted at its centre, and embeds. Find $\omega$.**

- angular momentum about the pivot is conserved; linear momentum is not (the pivot exerts a force)
- before: $L = mvr = (0.020)(300)(0.30) = 1.8\ \mathrm{kg\,m^2/s}$
- after: $I = \frac{1}{12}Ml^2 + mr^2 = 0.045 + 0.0018 = 0.0468$
- $\omega = L/I = 38$ rad/s

Choose angular momentum whenever there is a pivot. The pivot's own force acts through the axis, so it exerts no torque and drops out.

## Pulling mass inward lowers I – one

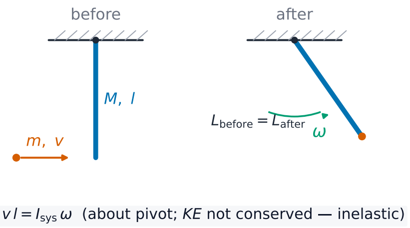



*Pulling mass inward lowers I, so $\omega$ rises to conserve L = I*

Pulling mass inward lowers I, so $\omega$ rises to conserve L = I



## Pulling mass inward lowers I – two




*Angular momentum about the pivot is conserved as the bullet embeds in the rod*

Angular momentum about the pivot is conserved as the bullet embeds in the rod

# 3 Rolling

## Rolling without slipping

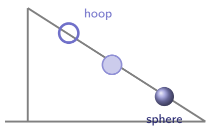
**Rolling without slipping** 无滑滚动 links translation and rotation:

**The condition**

$$v_{cm} = \omega r, \qquad a_{cm} = \alpha r$$

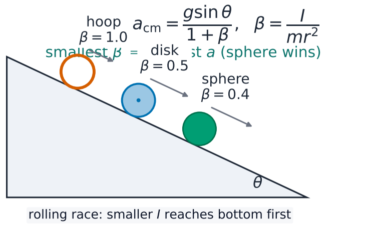
**Static friction** 静摩擦 produces the rolling but does no work (the contact point is momentarily at rest).

> Down a ramp: the solid sphere wins, the hoop last.






## Rolling race: the shape with the smallest – one




*Rolling race: the shape with the smallest $I/mr^2$ reaches the bottom first*

> Rolling race: the shape with the smallest $I/mr^2$ reaches the bottom first

## Where the energy goes when a shape rolls

| | |
|---|---|
| **the split** | a rolling body has both translational and rotational kinetic energy |
| **write $I = \beta mr^2$** | $\beta$ is the shape factor: hoop 1, disc $\frac{1}{2}$, solid sphere $\frac{2}{5}$, spherical shell $\frac{2}{3}$ |
| **racing down an incline** | $a_{cm} = \frac{g\sin\theta}{1+\beta}$, from $mgh = \frac{1}{2}mv^2(1+\beta)$ |
| **so** | $v = \sqrt{\frac{2gh}{1+\beta}}$ – mass and radius both cancel |
| **the winner** | the smallest $\beta$: solid sphere, then disc, then shell, then hoop |
| **what $\beta$ measures** | how much **rotational inertia** 转动惯量 sits far from the axis |

> Mass and radius cancel, so the race is decided by shape alone – and $\Delta x_{cm} = r\Delta\theta$ ties the centre's displacement to the angular displacement.



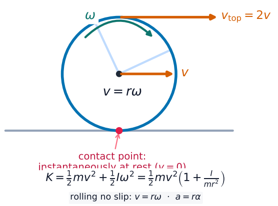
## The rolling race, decided by one number

| | |
|---|---|
| **write $I = \beta mr^2$** | $\beta$ is the shape factor: hoop 1, disc $\frac{1}{2}$, solid sphere $\frac{2}{5}$, shell $\frac{2}{3}$ |
| **energy conservation** | $mgh = \frac{1}{2}mv^2(1+\beta)$, so $v = \sqrt{\frac{2gh}{1+\beta}}$ |
| **so the winner** | the smallest $\beta$ – the solid sphere, then the disc, then the shell, then the hoop |
| **and the mass?** | cancels completely: size and mass do not matter, only the shape |

> $\beta$ measures how much mass sits far from the axis. Mass at the rim has to be spun up as well as moved forward, and that energy is taken from the translation.



## Rolling race: the shape with the smallest – two




*In rolling without slipping the contact point is at rest, so $v = r$*

> In rolling without slipping the contact point is at rest, so $v = r$

# 4 Satellites

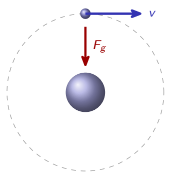
## Orbiting satellites

Gravity supplies the centripetal force. Set them equal:

**Orbital speed and Kepler**

$$v = \sqrt{\frac{GM}{r}}, \qquad T^2 \propto r^3$$

- energy $E = K + U$, with $U = -GMm/r$
- elliptical orbits conserve energy *and* angular momentum




A bigger orbit means a slower speed, longer period.

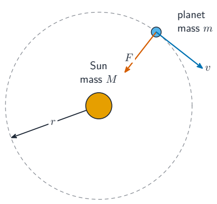
## The energy of an orbit is negative, and that matters

| | |
|---|---|
| **gravitational potential energy 引力势能** | $U_g = -\frac{GMm}{r}$, taking zero at infinity |
| **a circular orbit** | $K = -\frac{1}{2}U$, so the **total mechanical energy 总机械能** is $E = K + U = -\frac{GMm}{2r}$ |
| **negative means bound** | the satellite cannot escape; raising it to a larger orbit means *increasing* $E$ toward zero |
| **escape velocity 逃逸速度** | set $E = 0$: $v_{esc} = \sqrt{\frac{2GM}{r}} = \sqrt{2}\times$ the circular orbital speed |
| **Kepler's second law 开普勒第二定律** | equal areas in equal times – which is conservation of angular momentum in disguise |

Larger orbits are slower ($v = \sqrt{GM/r}$) yet carry more energy. Speeding a satellite up moves it to a higher, slower orbit – the counter-intuitive result AP loves to test.



## Motion of Orbiting Satellites




Gravity provides the centripetal force that keeps a satellite in orbit



## Worked example – a satellite in low Earth orbit

**A satellite orbits at $r = 7.0 \times 10^6$ m, with $GM = 4.0 \times 10^{14}$ m$^3$/s$^2$. Find its speed and escape speed.**

- $v = \sqrt{GM/r} = \sqrt{4.0 \times 10^{14}/7.0 \times 10^6} = 7.6 \times 10^3$ m/s
- $v_{esc} = \sqrt{2}\, v = 1.1 \times 10^4$ m/s
- period $T = 2\pi r/v \approx 5.8 \times 10^3$ s, about 97 minutes

**Exam skill.** Orbit FRQs are energy-and-angular-momentum problems in disguise. Write $E = \frac{1}{2}mv^2 - \frac{GMm}{r}$ and $L = mvr\sin\theta$ at the two points of interest and solve the pair – never assume the circular-orbit formulae apply to an ellipse.



## Exam tips

- Compute the **moment of inertia** $I = \int r^2\, dm$ and shift axes with the **parallel-axis theorem** $I = I_{cm} + Md^2$.
- Rotational kinetic energy is $\frac{1}{2}I\omega^2$; a rolling body has both translational and rotational KE.
- Conserve **angular momentum** $L = I\omega$ when net external torque is zero (a spinning skater pulling in).
- Use energy conservation for rolling-without-slipping problems ($v = r\omega$ ties the two motions).
- Know standard $I$ values (hoop, disk, rod, sphere) and where the axis is.

# 5 Recap

## Unit 6 – key takeaways

**1 Rotational energy**

$K_{rot} = \frac{1}{2} I\omega^2$; $W = \int \tau \, d\theta$

**2 Angular momentum**

$\vec{L} = \vec{r} \times \vec{p} = I\omega$; $\vec{\tau} = d\vec{L}/dt$

**3 Conservation**

no external torque $\Rightarrow I_i\omega_i = I_f\omega_f$

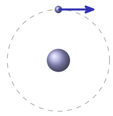
**4 Orbits**

$v = \sqrt{GM/r}$, Kepler's $T^2 \propto r^3$



## Check yourself

1. Write the net torque as a rate of change of angular momentum.
2. A skater pulls their arms in – what happens to $\omega$?
3. Does the static friction on a rolling wheel do work?
4. A satellite moves to a bigger orbit – faster or slower?



**Answers** **1.** $\vec{\tau} = d\vec{L}/dt$ **2.** it speeds up **3.** no **4.** slower