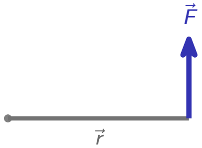
Torque & Rotational Dynamics

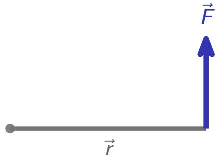
AP Physics C: Mechanics ▪ Unit 5

AP Physics C: Mechanics



What we will cover

1. Rotational kinematics
2. Linking linear and rotational motion
3. Torque as a cross product
4. Rotational inertia
5. Rotational equilibrium
6. Newton's second law for rotation



1 Rotational kinematics

Rotational kinematics

Angular motion mirrors the linear case – through calculus:

Derivatives

$$\omega = \frac{d\theta}{dt}, \qquad \alpha = \frac{d\omega}{dt}$$

Constant-α equations have the same form as SUVAT; for a variable α, integrate: $\omega = \int \alpha \, dt$.

Worked example ($\alpha = 4t$)

$$\omega = \int_0^t 4t' \, dt' = 2t^2$$

$$\theta = \int_0^t 2t'^2 \, dt' = \tfrac{2}{3} t^3$$

Only calculus handles a non-constant α.

Worked example – a fan blade slowing to rest

A fan blade slows from $\omega_0 = 30$ rad/s to rest with $\alpha = -6.0$ rad/s^2. How long, and through what angle?

- $\omega = \omega_0 + \alpha t \Rightarrow 0 = 30 - 6.0t$, so $t = 5.0$ s
- $\omega^2 = \omega_0^2 + 2\alpha\,\Delta\theta \Rightarrow \Delta\theta = \dfrac{30^2}{2(6.0)}$
- $= 75$ rad, about 12 revolutions

Identical in form to a car braking from 30 m/s at 6.0 m/s^2. Every rotational kinematics problem is a linear one you have already solved, with the symbols swapped.

The rotational quantities, and their radian rule

angular displacement 角位移 θ	in **radians 弧度** – AP works in radians throughout, so degrees must be converted first
angular velocity 角速度 ω	$\omega = d\theta/dt$, in rad/s
angular acceleration 角加速度 α	$\alpha = d\omega/dt$, in rad/s^2
linking to linear	$v = \omega r$, $a_t = \alpha r$ (**tangential 切向**), $a_c = \omega^2 r$ (**centripetal 向心**)
the equations	the SUVAT set carries over symbol for symbol, for constant α only

> $s = r\theta$ is true only in radians. Every one of those linking formulas fails silently if a degree slips in.

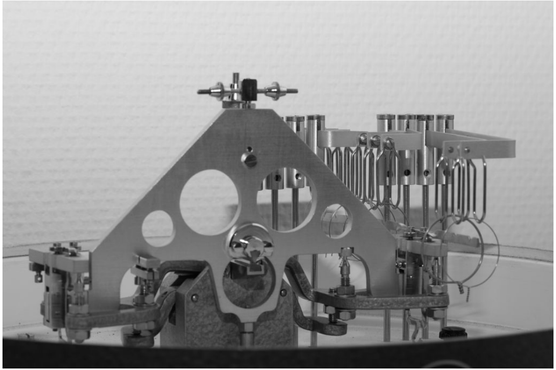
Linking linear and rotational motion

Every point on a **rigid body** 刚体 moves along a curve, tied to the radius r:

Same ω, different v

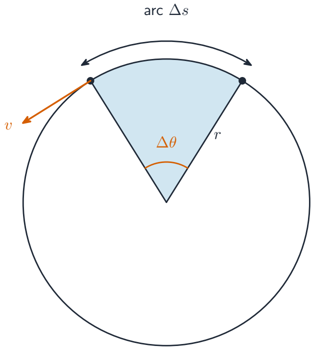
$$v = \omega r, \qquad a_t = \alpha r$$

All points share one ω; points farther out move faster.



Balance scale

As the radius turns through an angle



As the radius turns through an angle, a point moves along an arc at speed v

Two accelerations of a point on a wheel

Centripetal acceleration	**Centripetal acceleration** $a_c = \omega^2 r$ points to the axis. It is present whenever the point is going round, spinning up or not.
Tangential acceleration	**Tangential acceleration** $a_t = \alpha r$ points along the motion. It is zero at constant ω.
Together	They are perpendicular, so the total is $\sqrt{a_c^2 + a_t^2}$ – and the direction turns towards the axis as ω grows.

> "Uniform" rotation means $a_t = 0$, never $a = 0$. The point is still accelerating, towards the centre.

2 Torque

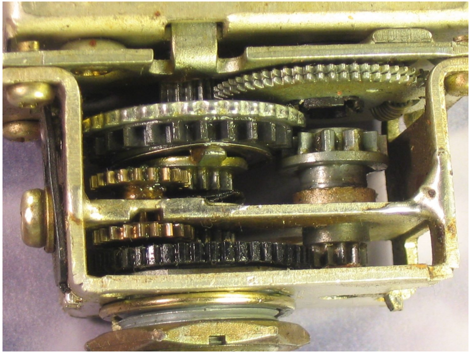
Torque

Torque 力矩 is the rotational effect of a force:

A cross product

$$\tau = rF\sin\theta, \qquad \vec{\tau} = \vec{r} \times \vec{F}$$

The **lever arm** 力臂 is the perpendicular distance to the force – a force farther out and $\perp$ to $\vec{r}$ twists most.



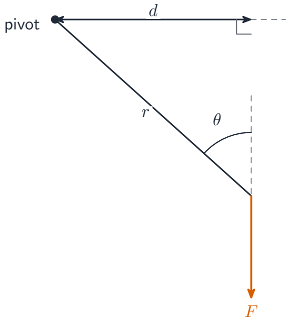
Gear train

Torque is a vector product

the definition	$\vec{\tau} = \vec{r} \times \vec{F}$, magnitude $rF \sin\theta$
the lever arm	only the component of $\vec{F}$ perpendicular to $\vec{r}$ turns anything – a force pointing at the pivot has zero torque
direction	perpendicular to the plane of $\vec{r}$ and $\vec{F}$, fixed by the **right-hand rule** 右手定则
why a vector product 矢量积	it captures both the size *and* the sense of the turn in one object

> Push a door at the hinge and nothing happens; push at the handle, perpendicular, and it swings. $\sin\theta$ and r are that sentence in symbols.

The torque of a force depends on



> The torque of a force depends on the perpendicular distance from the axis

3 Rotational inertia

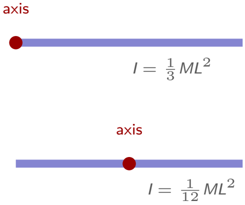
Rotational inertia

Rotational inertia 转动惯量 resists angular acceleration, and depends on how mass is spread:

Sum, or integrate

$$I = \sum m_i r_i^2 = \int r^2\, dm$$

Parallel-axis theorem 平行轴定理: $I = I_{cm} + Md^2$.



A rod is harder to spin about its end.

Building a rotational inertia by integration

the definition	$I = \int r^2\, dm$ – r is the distance from the axis, not from the centre
the mass element	use the right density: $dm = \lambda\, dl$ for a rod (**linear density 线密度**), $\sigma\, dA$ for a plate, $\rho\, dV$ for a solid
coaxial shells	build a solid cylinder from thin shells of radius r, thickness dr: $dm = \rho(2\pi rL)\, dr$
the same method	handles cylindrical shells and annular rings – change only the limits
parallel axis	$I = I_{\mathrm{cm}} + Md^2$ for an axis a distance d off the centre of mass

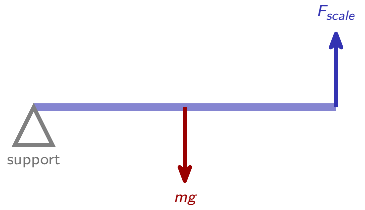
I depends on where the axis is, not just on the object. The same rod has $\frac{1}{12}ML^2$ about its centre and $\frac{1}{3}ML^2$ about its end.

4 Equilibrium

Rotational equilibrium

In **rotational equilibrium** 转动平衡 the **net torque** 净力矩 is zero. Full **static equilibrium** 静平衡: both net force and net torque zero.

- draw an **extended free-body diagram** 扩展受力图 (where each force acts)
- put the axis at an unknown force to drop it out



$$mg \times 2 = F_{scale} \times 4 \Rightarrow F_{scale} = 49\,\mathrm{N}.$$

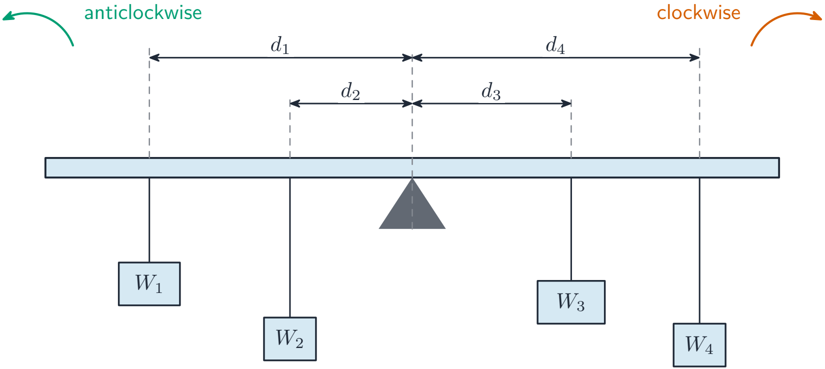
Worked example – a beam on a pivot

A uniform 20 kg beam, 4.0 m long, is pivoted at its left end and held horizontal by a vertical rope at its right end. Find the rope tension.

- take torques about the pivot, so its unknown force drops out
- weight acts at the centre: $\tau_W = (20)(9.8)(2.0) = 392$ N m
- rope: $T(4.0) = 392$, so $T = 98$ N

Choosing the pivot as the axis is the whole technique: any unknown force *through* your axis contributes no torque, so it never enters the equation.

At balance the clockwise and anticlockwise torques



At balance the clockwise and anticlockwise torques about the axis are equal

5 Rotational Newton

Newton's second law for rotation

A nonzero net torque gives an angular acceleration – exactly like $F = ma$:

Rotational form

$$\tau_{net} = I\alpha$$

torque $\leftrightarrow$ force, $I \leftrightarrow$ mass, $\alpha \leftrightarrow$ acceleration.

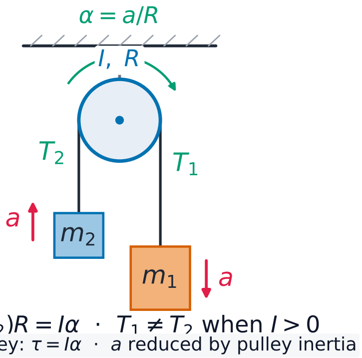
Worked example

$\tau = 6\,\mathrm{N\,m}$ on $I = 2\,\mathrm{kg\,m^2}$:

$$\alpha = \frac{6}{2} = 3\,\mathrm{rad/s^2}$$

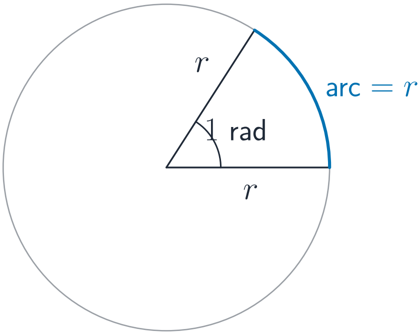
Double $I \Rightarrow$ half the α. For a pulley, pair it with $F = ma$ (linked by $a = \alpha r$).

A massive pulley



A massive pulley: the two tensions differ because the pulley needs net torque

Rotational Kinematics, continued



one **radian**: the angle whose arc is exactly one radius long

π rad $= 180°$, so 1 rad $\approx 57.3°$

degrees $\rightarrow$ radians: $\times \frac{\pi}{180}$ radians $\rightarrow$ degrees: $\times \frac{180}{\pi}$

One radian is the angle whose arc length equals the radius

Rotational FRQs score in the setup

1. separate diagrams	one free-body diagram for each mass *and* one for the pulley – the pulley's is the one candidates omit
2. state the constraint	$a = R\alpha$, linking the string's linear acceleration to the pulley's rotation
3. state the sign convention	which way is positive for rotation, and keep it for every equation
4. then solve	e.g. with $I/R^2 = 12.5$ kg: $a = \dfrac{(m_1 - m_2)g}{m_1 + m_2 + I/R^2} = \dfrac{19.6}{18.5} = 1.1\ \mathrm{m/s^2}$, giving $T_1 = m_1(g - a) = 35$ N

A massive pulley makes the two tensions different – that difference is exactly what supplies the net torque that spins it up.

Exam tips

- Use the rotational analogues: $\theta, \omega = \frac{d\theta}{dt}, \alpha = \frac{d\omega}{dt}$, with the constant-$\alpha$ equations mirroring linear kinematics.
- Convert between linear and angular with $v = r\omega$ and $a_t = r\alpha$ (plus centripetal $a_c = \frac{v^2}{r} = \omega^2 r$).
- Keep radians throughout and fix a positive sense of rotation.
- Relate torque to angular acceleration through $\tau = I\alpha$.
- Draw the rotation axis – the moment arm is the perpendicular distance to it.

6 Recap

Unit 5 – key takeaways

1 Kinematics

$\omega = d\theta/dt$, $\alpha = d\omega/dt$; integrate for variable α

2 Torque

$\vec{\tau} = \vec{r} \times \vec{F}$; the lever arm sets the twist

3 Inertia

$I = \int r^2\, dm$; mass far from the axis counts most

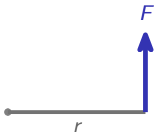
4 Dynamics

equilibrium: net $\tau = 0$; otherwise $\tau = I\alpha$

Check yourself

1. What integral gives the rotational inertia of a continuous body?

2. Is a rod harder to spin about its end or its center?



3. Torque is the cross product of which two vectors?

4. $\tau = 10\,\mathrm{N\,m}$ on $I = 5\,\mathrm{kg\,m^2}$ – find α.

Answers **1.** $\int r^2\,dm$ **2.** its end **3.** $\vec{r}$ and $\vec{F}$ **4.** $2\,\mathrm{rad/s^2}$