Energy and Momentum of Rotating Systems

AP Physics C: Mechanics

Rotational Kinetic Energy

A spinning body has **rotational kinetic energy** 转动动能

$$K_{\text{rot}} = \tfrac{1}{2} I\omega^2,$$

the rotational twin of $\frac{1}{2}mv^2$, with the **rotational inertia** 转动惯量 I playing the role of mass. A body that both moves and spins carries **both** terms:

$$K = \tfrac{1}{2} m v_{\text{cm}}^2 + \tfrac{1}{2} I\omega^2.$$

Worked example. A uniform cylinder ($I = \frac{1}{2}mr^2$) rolls at speed v. Its kinetic energy is $K = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{v}{r}\right)^2 = \frac{3}{4}mv^2$ – one third of it is rotational. The same ball of energy bookkeeping decides every rolling problem.

Torque and Work

A **torque** 力矩 acting through an **angular displacement** 角位移 does work, and power is torque times angular velocity:

$$W = \int_{\theta_1}^{\theta_2} \tau\, d\theta, \qquad P = \tau\omega.$$

These are the rotational forms of $W = \int F\,dx$ and $P = Fv$ – the whole translational energy toolkit carries over with $F \to \tau$, $x \to \theta$, $v \to \omega$.

Worked example. A motor applies a constant 8.0 N · m torque to a flywheel for 5.0 full turns: $W = \tau\,\Delta\theta = 8.0(5.0)(2\pi) = 250$ J, which appears as rotational kinetic energy if friction is negligible.

Angular Momentum and Angular Impulse

Angular momentum 角动量 for a particle is

$$\vec{L} = \vec{r} \times \vec{p}, \qquad |L| = mvr\sin\theta,$$

so even a particle moving in a straight line has angular momentum about any point not on that line ($L = mv\,d$, with d the perpendicular distance). For a rigid body spinning about a fixed axis, $L = I\omega$. Newton's second law in rotational form is

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} \quad (= I\alpha \text{ when } I \text{ is constant}),$$

and a net torque acting over time delivers an **angular impulse** 角冲量 $\int \tau\,dt = \Delta L$ – the rotational impulse–momentum theorem.

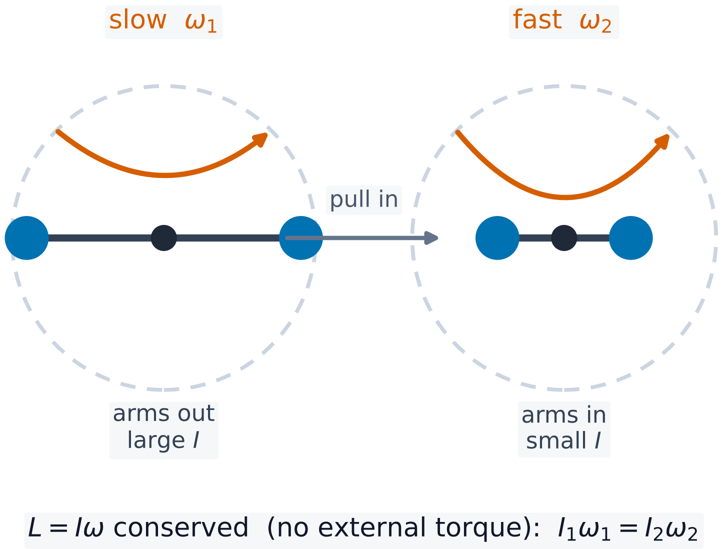
Conservation of Angular Momentum

With **zero net external torque**, total angular momentum is **conserved** 守恒:

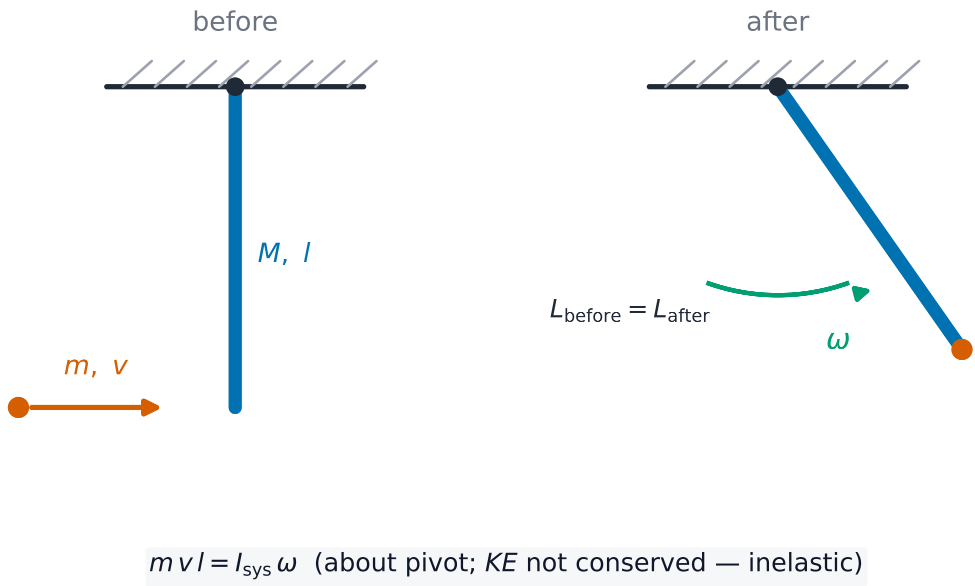
$$L_{\text{before}} = L_{\text{after}}, \qquad I_1\omega_1 = I_2\omega_2 \text{ (one rigid body reshaping).}$$

If I shrinks, ω grows – a spinning skater speeds up pulling her arms in. The rule survives collisions and shape changes, which is what makes it so useful: pick the axis so that every external force (gravity, the pivot force) exerts no torque about it.

Worked example. A skater spins at 2.0 rev/s with $I_1 = 4.0\ \text{kg}\cdot\text{m}^2$. Pulling in her arms drops it to $I_2 = 1.6\ \text{kg}\cdot\text{m}^2$: $\omega_2 = \dfrac{4.0}{1.6}(2.0) = 5.0$ rev/s. Her kinetic energy *rises* – the extra energy is the work her muscles do pulling her arms inward.



Pulling mass inward lowers I, so ω rises to conserve $L = I\omega$



Angular momentum about the pivot is conserved as the bullet embeds in the rod

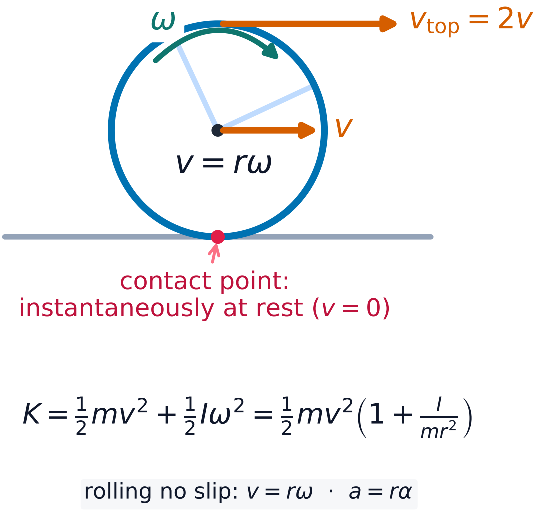
Worked example (rotational collision). A 0.020 kg bullet at 300 m/s strikes the tip of a uniform rod ($M = 1.5$ kg, length $l = 0.60$ m) hanging from a pivot, and embeds. About the pivot, gravity and the pivot force exert no torque during the strike, so L is conserved: $L = mvl = 0.020(300)(0.60) = 3.6\ \text{kg}\cdot\text{m}^2/\text{s}$. Afterwards $I = \frac{1}{3}Ml^2 + ml^2 = 0.18 + 0.0072 = 0.187\ \text{kg}\cdot\text{m}^2$, so $\omega = \dfrac{3.6}{0.187} \approx 19$ rad/s. (Linear momentum is *not* conserved here – the pivot pushes on the rod – and kinetic energy certainly is not: check both before claiming them.)

Figure skating and angular momentum: a skater who pulls their arms in during a spin lowers their rotational inertia, so they spin faster

Rolling

Rolling without slipping 无滑滚动 ties translation to rotation: the contact point is momentarily at rest, so

$$\Delta x_{\rm cm} = r\,\Delta\theta, \qquad v_{\rm cm} = r\omega, \qquad a_{\rm cm} = r\alpha.$$



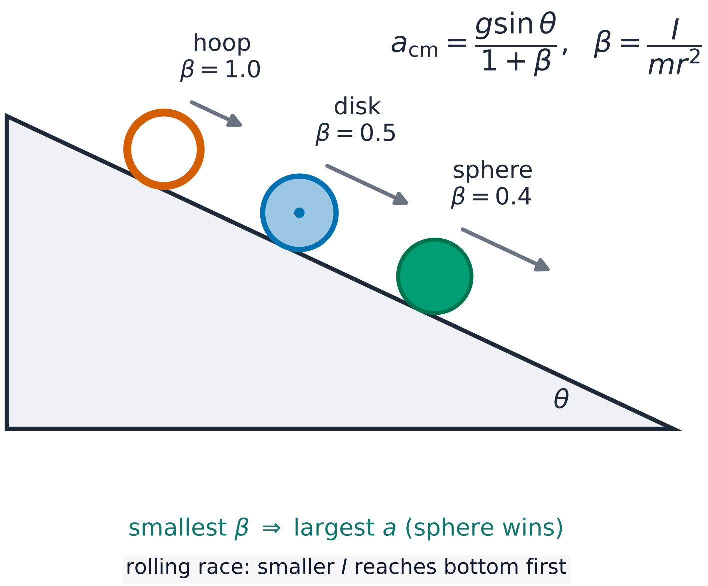
In rolling without slipping the contact point is at rest, so $v = r\omega$

Static friction supplies the torque that keeps the spin matched to the motion, but at a point that is not sliding – so for pure rolling, friction does **no work**, and energy conservation is safe to use. (Rolling friction is beyond the AP course.)

Racing shapes down an incline shows the energy split. With $I = \beta m r^2$, energy conservation gives

$$a_{\text{cm}} = \frac{g \sin\theta}{1+\beta}:$$

a **sphere** ($\beta = \frac{2}{5}$) beats a **disk** 圆盘 ($\beta = \frac{1}{2}$), which beats a **hoop** 圆环 ($\beta = 1$) – mass and radius cancel completely. More of the hoop's energy is locked in rotation, so its center moves slower.



Rolling race: the shape with the smallest I/mr^2 reaches the bottom first

Worked example. A solid sphere rolls from rest down a 30° incline: $a = \dfrac{g\sin 30°}{1+\frac{2}{5}} = \dfrac{9.8(0.50)}{1.4} = 3.5\ \mathrm{m/s}^2$, versus $4.9\ \mathrm{m/s}^2$ for a frictionless slider – rolling objects always lose the race against sliding ones.

Motion of Orbiting Satellites

Gravity supplies the **centripetal force 向心力** for a **satellite 卫星** in a circular orbit:

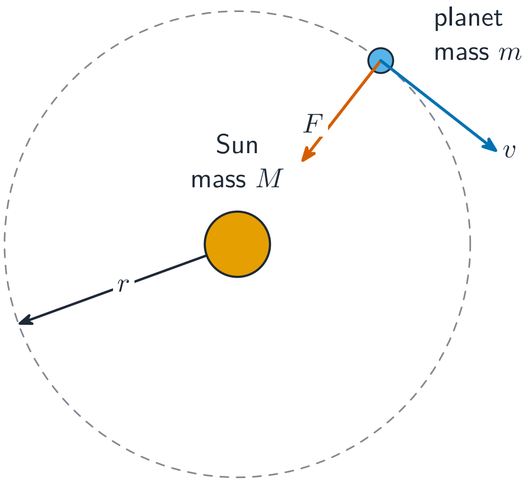
$$\frac{GMm}{r^2} = \frac{mv^2}{r} \quad \Rightarrow \quad v = \sqrt{\frac{GM}{r}}$$

– larger orbits are slower. With **gravitational potential energy 引力势能** $U_g = -\dfrac{GMm}{r}$, a circular orbit obeys $K = -\frac{1}{2}U$, so the **total mechanical energy 总机械能** is

$$E = K + U = \frac{U}{2} = -\frac{GMm}{2r},$$

negative because the satellite is bound. To escape from radius r, total energy must reach zero, giving the **escape velocity 逃逸速度**

$$v_{\text{esc}} = \sqrt{\frac{2GM}{r}}.$$



Gravity provides the centripetal force that keeps a satellite in orbit

In an **elliptical orbit** 椭圆轨道, E and L are fixed: gravity points at the focus, so it exerts no torque about it. Conserved L means the satellite sweeps equal areas in equal times – **Kepler's second law** 开普勒第二定律 – and therefore moves fastest at closest approach, slowest at the far point. Energy conservation connects speeds at the two ends.

Worked example. For a satellite at $r = 7.0 \times 10^6$ m around Earth ($GM = 4.0 \times 10^{14}$ m^3/s^2): orbital speed $v = \sqrt{GM/r} = 7.6 \times 10^3$ m/s, while escaping from that radius needs $v_{\text{esc}} = \sqrt{2GM/r} = 1.1 \times 10^4$ m/s – exactly $\sqrt{2}$ times the circular speed.

Exam skill. Orbit FRQs are energy-and-angular-momentum problems in disguise: write $E = \frac{1}{2}mv^2 - \dfrac{GMm}{r}$ and $L = mvr\sin\theta$ at the two points of interest and solve the pair – never assume the orbit formulae for circles apply to an ellipse.

Exam tips

- Compute the **moment of inertia** $I = \int r^2\,dm$ and shift axes with the **parallel-axis theorem** $I = I_{cm} + Md^2$.
- Rotational kinetic energy is $\frac{1}{2}I\omega^2$; a rolling body has both translational and rotational KE.
- Conserve **angular momentum** $L = I\omega$ when net external torque is zero (a spinning skater pulling in).

- Use energy conservation for rolling-without-slipping problems ($v = r\omega$ ties the two motions).
- Know standard I values (hoop, disk, rod, sphere) and where the axis is.