# Infinite Sequences & Series

## AP Calculus BC ▪ Unit 10

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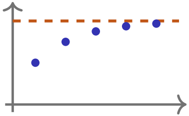
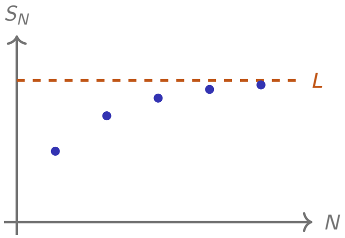


# What we will cover

1. Convergence & partial sums
2. Geometric series
3. The convergence tests
4. Absolute vs conditional
5. Taylor polynomials & error
6. Power series & intervals



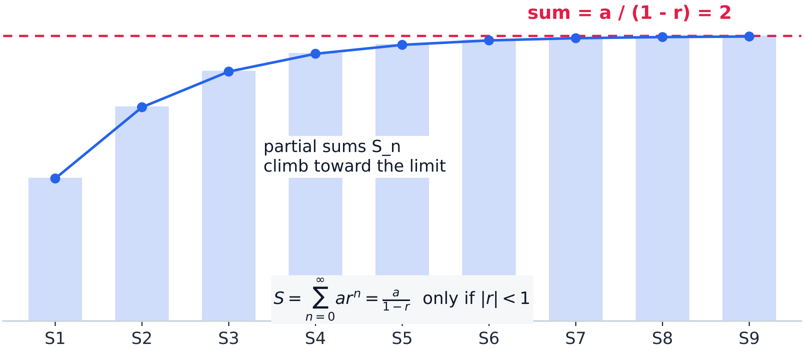
# 1 Convergence

# Convergent and divergent series

An **infinite series** 无穷级数 $\sum a_n$ is defined as the limit of its **partial sums** 部分和 $S_N = a_1 + \cdots + a_N$.

> Every convergence question is really a question about the limit of the partial sums.

# Convergent and divergent series, in detail

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# Geometric series – the one you can sum exactly

A **geometric series** 几何级数 has a constant ratio $r$. It converges exactly when $|r| < 1$:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

> It underlies power series later in the unit.

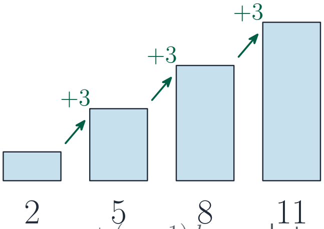
**Sum $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots$**

Here $a = 3$ and $r = \frac{1}{2}$, and $|r| < 1$:

$$\frac{a}{1-r} = \frac{3}{1-\frac{1}{2}} = 6.$$

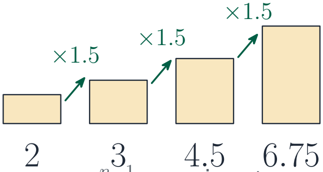
# Working with Geometric Series

**arithmetic**: add $d$ each step




$u_n = a + (n-1)d$: equal steps

**geometric**: multiply by $r$ each step




$u_n = ar^{n-1}$: growing steps

A geometric sequence multiplies by the same ratio at each step

# 2 The tests

# Start here – the *n*th term test

If the terms do not shrink to zero, the sum cannot settle:

$$\lim_{n\to\infty} a_n \neq 0 \implies \text{diverges.}$$

Always check this quick test first.

It is **only** a test for divergence

If $a_n \to 0$ the test is **inconclusive** – the series may still diverge

# $p$-series and the harmonic series

$$\sum \frac{1}{n^p} \begin{cases} \text{converges} & p > 1 \\ \text{diverges} & p \leq 1 \end{cases}$$

The **integral test** 积分判别法 proves this: for positive, decreasing, continuous $f$, $\sum a_n$ and $\int_1^\infty f$ both converge or both diverge.

> The **harmonic series** 调和级数 $\sum \frac{1}{n}$ ($p = 1$) diverges – even though its terms go to zero. A famous, must-know fact.

The $p$-series family is the standard yardstick for the comparison tests.

# Comparison and alternating series

**Direct comparison 直接比较**

$0 \le a_n \le b_n$ and $\sum b_n$ converges $\Rightarrow$ so does $\sum a_n$

**Limit comparison 极限比较**

$\lim \frac{a_n}{b_n}$ finite and positive $\Rightarrow$ both do the same thing

**Alternating series 交错级数**

$b_n$ positive, decreasing, and $b_n \to 0 \Rightarrow$ converges

**Ratio test 比值判别法**

$L = \lim \left|\frac{a_{n+1}}{a_n}\right|$: $L < 1$ converges, $L > 1$ diverges, $L = 1$ inconclusive

# Worked example – the ratio test

**Test $\sum \frac{n}{2^n}$**

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} \to \frac{1}{2} < 1$$

so the series converges.

The ratio test is the go-to for factorials or $n$th powers – and it is exactly how you find a radius of convergence.

# Absolute vs conditional convergence

**Absolutely 绝对收敛**
$\sum |a_n|$ converges –
the stronger property

**Conditionally 条件收敛**
$\sum a_n$ converges but
$\sum |a_n|$ does not

The classic: $\sum \frac{(-1)^n}{n}$ converges conditionally – it relies on the cancellation of signs, since $\sum \frac{1}{n}$ diverges.

**Alternating error bound**

$$|S - S_N| \leq b_{N+1}$$

The error is no larger than the first omitted term.

# Absolute against conditional convergence

| | |
|---|---|
| **Converges absolutely** | A series **converges absolutely** when $\sum \lvert a_n \rvert$ converges – and then $\sum a_n$ converges too, whatever the signs do. |
| **Converges conditionally** | It **converges conditionally** when $\sum a_n$ converges but $\sum \lvert a_n \rvert$ does not – the alternating harmonic series is the standard case. |
| **Why it matters** | An absolutely convergent series can be rearranged freely; a conditionally convergent one can be rearranged to give *any* sum at all. |
| **How to test** | Test $\sum \lvert a_n \rvert$ first. If it converges you are finished; if not, try the alternating series test on $\sum a_n$. |

"Converges" is not one property. Say which kind, because the error bound and the rearrangement both depend on it.

# 3 Taylor

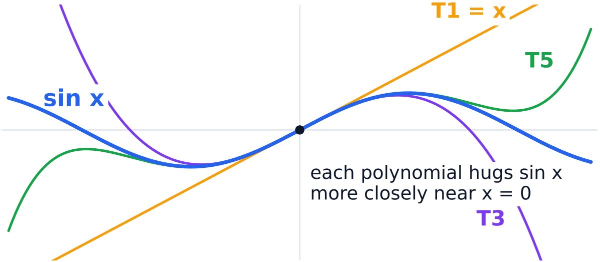
# Taylor polynomials

A **Taylor polynomial** 泰勒多项式 approximates $f$ near a centre $a$ using its derivatives there:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

> Each added term matches one more derivative. Centred at $a = 0$ it is a Maclaurin polynomial.

# Taylor polynomials, in detail




$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \quad \cdot \quad \text{for}$$

$$\sin x\colon\ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

> Each added term matches one more derivative at the centre, so the polynomial hugs the curve more closely – and only near $a$.

# The Lagrange error bound

How far can a Taylor polynomial be from the truth?

$$|R_n(x)| \leq \frac{\max \left| f^{(n+1)}(z) \right|}{(n+1)!} \left| x - a \right|^{n+1}.$$

Bound the $(n+1)$th derivative on the interval, then compute – the standard way to prove an estimate is accurate enough.

# Radius and interval of convergence

A **power series** 幂级数 $\sum c_n(x-a)^n$ converges within a **radius** 收敛半径 $R$ of its centre. Find $R$ with the ratio test, then test the endpoints separately.

> The ratio test is inconclusive at the endpoints – that is why they need their own check.

**Interval of $\sum \frac{x^n}{n}$**

Ratio: $|x|\frac{n}{n+1} \to |x| < 1$, so $R = 1$. $x = -1$: alternating harmonic – converges.
$x = 1$: harmonic – diverges.

$$\text{Interval} = [-1, 1).$$

# The four Maclaurin series to memorise

$$e^x = \sum \frac{x^n}{n!}$$

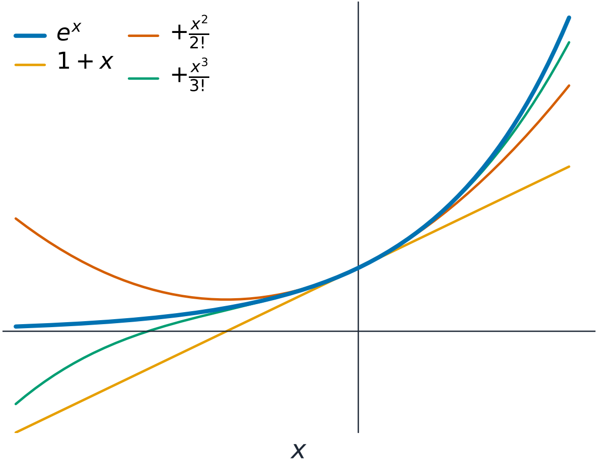
$$\frac{1}{1-x} = \sum x^n$$

$$\sin x = \sum \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum \frac{(-1)^n x^{2n}}{(2n)!}$$

New series come from manipulating these – substituting, differentiating, integrating, multiplying.

# The four Maclaurin series to memorise, in detail




Each extra Maclaurin term hugs the function over a wider range

# Worked example – the substitution trick

## Maclaurin series for $e^{x^2}$

Substitute $x^2$ for $x$ in $e^x = \sum \frac{x^n}{n!}$:

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

which converges for all $x$.

This substitution is far faster than differentiating $e^{x^2}$ six times – which is exactly what the exam is testing.

# Defining Convergent and Divergent Infinite Series

> Nested matryoshka dolls: an infinite series keeps adding terms – some converge, some diverge

# Exam tips

- Test a series for convergence with the right tool: **geometric** ($|r| < 1$, sum $\frac{a}{1-r}$), $n$th-term, ratio, integral, comparison, or alternating-series test.
- A **geometric** infinite sum converges only when $|r| < 1$; otherwise it diverges.
- Build a **Taylor/Maclaurin** series to approximate a function; more terms give a better fit **near the centre**.
- Know the standard Maclaurin series for $e^x$, $\sin x$, $\cos x$, and $\frac{1}{1-x}$.
- Find the **radius/interval of convergence** with the ratio test, then check the endpoints separately.

# 4 Recap

# Unit 10 – key takeaways

1 **Convergence**

the limit of the partial sums; check $a_n \to 0$ first

2 **Geometric**

$|r| < 1 \Rightarrow a/(1 - r)$ – the one exact sum

3 **The tests**

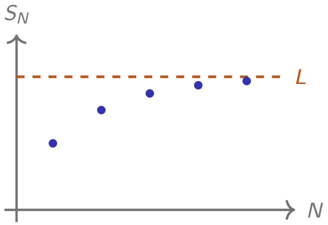
$p$-series yardstick; comparison, alternating, ratio

4 **Taylor**

$P_n$ matches derivatives at $a$; ratio test gives $R$

# Check yourself

1. Does $\sum \frac{1}{n}$ converge?
2. Sum $3 + \frac{3}{2} + \frac{3}{4} + \cdots$
3. The ratio test gives $L = 1$. What follows?
4. Give the Maclaurin series for $e^{x^2}$.




partial sums

**Answers** **1.** no – it diverges **2.** $6$ **3.** inconclusive – use another test **4.** $\sum x^{2n}/n!$