Differential Equations

AP Calculus BC ▪ Unit 7

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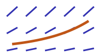


What we will cover

$\frac{dy}{dt} = ky$

↓

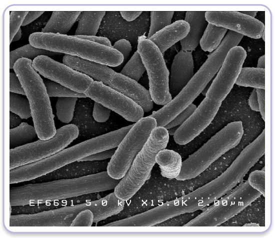
$y = y_0 e^{kt}$

1 Modelling

Modelling a situation with a rate

A **differential equation** 微分方程 relates a function to its own derivatives – it describes a situation by its rate of change. "The rate of change of a quantity is proportional to its size" becomes

$$\frac{dy}{dt} = ky.$$



Bacteria growth

Verifying a proposed solution

A **solution** 解 is a function that makes the equation true. To verify one: differentiate it and substitute – if both sides match, it is a solution.

A differential equation usually has infinitely many solutions – a whole family, differing by a constant.

Check $y = 2e^{x^2/2}$ **solves** $\frac{dy}{dx} = xy$

$$\frac{dy}{dx} = 2e^{x^2/2} \cdot x = x\left(2e^{x^2/2}\right) = xy.$$

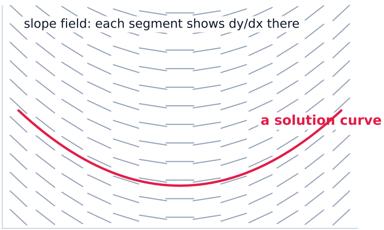
True.

2 Slope fields

Slope fields

A **slope field** 斜率场 draws the equation as a grid of short segments: at each (x, y) the segment has slope $\frac{dy}{dx}$ there.

> It shows the shape of the solution curves without solving.



Approximating Solutions Using Euler's Method

Euler's method 欧拉方法 approximates a solution numerically by stepping along the slope field.

> Starting from a known point, take a small step Δx and update:
>
> $$y_{\text{new}} = y_{\text{old}} + \frac{dy}{dx} \cdot \Delta x.$$
>
> Repeat.

Reading a slope field

- segments **flat** $\Rightarrow \frac{dy}{dx} = 0$: the solution is momentarily level
- segments **steepen** $\Rightarrow$ the solution rises or falls faster
- a **horizontal row** of equal slopes $\Rightarrow$ the rate depends only on y
- **Exam skill:** start at the given point and follow the segments

> A solution curve stays tangent to the field everywhere.

3 Separation of variables

Separation of variables

If $\frac{dy}{dx}$ factors into a function of x times a function of y, split them and integrate:

$$\frac{dy}{dx} = g(x)h(y) \implies \int \frac{dy}{h(y)} = \int g(x)\, dx.$$

> Add the constant of integration once – on the x-side.

all y one side, all x the other
↓
integrate both sides
↓
the **general solution** 通解

Worked example – separate, integrate, then pin it down

Solve $\frac{dy}{dx} = xy$ with $y(0) = 2$

$$\int \frac{dy}{y} = \int x\, dx \Rightarrow \ln|y| = \frac{x^2}{2} + C$$

$$\Rightarrow y = Ae^{x^2/2}$$

Applying $y(0) = 2$ gives $A = 2$:

$$y = 2e^{x^2/2}.$$

> An **initial condition** 初始条件 (x_0, y_0) picks one curve from the family – there is exactly one solution through a given point.

From the family to the particular solution

The constant gives a family of curves; the initial condition selects the **particular solution** 特解.

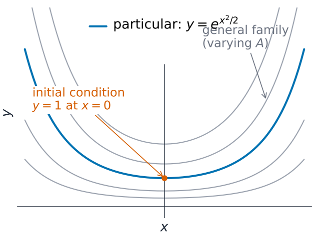
> Watch **domain restrictions** 定义域限制: keep the branch containing the initial point (e.g. the right sign of a square root).

Euler's method, and what it is good for

the idea	step along the slope field: $y_{n+1} = y_n + f(x_n, y_n)\,\Delta x$
the exam skill	carry out two or three steps by hand from a table – that is what is asked
the tabulation	columns for x, y, the slope, and Δy; one row per step
what it produces	an approximation, not a solution
the error	Euler's method underestimates a concave-up solution and overestimates a concave-down one
smaller steps	reduce the error, at the cost of more arithmetic

> The concavity argument is the same one as for a tangent-line approximation – and for the same reason: both follow a straight line away from a curving one.

From the family to the particular solution, in detail



> The constant gives a family of curves; the initial condition selects the **particular solution** 特解.

4 Exponential models

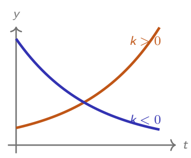
Exponential growth and decay

The most important model: rate proportional to size.

$$\frac{dy}{dt} = ky \implies y = y_0 e^{kt}$$

where y_0 is the value at $t = 0$.

> $k > 0$ gives **growth** 增长; $k < 0$ gives **decay** 衰减.



The exponential model, and where it fails

The equation	**Exponential growth or decay** is the solution of $\frac{dy}{dt} = ky$: rate proportional to amount, giving $y = y_0 e^{kt}$.
Reading k	$k > 0$ grows, $k < 0$ decays, and $\ln 2/\lvert k\rvert$ is the doubling or half-life.
Where it fails	Nothing grows proportionally forever. The logistic model adds the factor $(1 - y/L)$, so growth slows as y approaches the carrying capacity L.
The tell	Fastest growth in a logistic model happens at $y = L/2$ – the inflection point of the curve.

> Read the sentence into the differential equation: "proportional to the amount present" *is* $dy/dt = ky$.

Worked example – radioactive decay and half-life

$\frac{dy}{dt} = -0.10\,y$ (years), $y_0 = 50$ g

The solution is $y = 50e^{-0.10t}$, so after 10 years

$$y = 50e^{-1} = 18.4 \text{ g}.$$

Half-life 半衰期: solve $25 = 50e^{-0.10t}$, i.e. $e^{-0.10t} = \frac{1}{2}$:

$$t = \frac{\ln 2}{0.10} = 6.9 \text{ years}.$$

> The half-life is independent of the starting amount – that is what makes it a useful constant.

5 Logistic growth

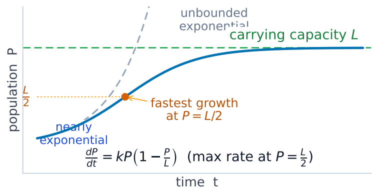
Logistic growth (BC only)

Real populations cannot grow for ever. The **logistic** 逻辑斯蒂 model adds a ceiling, the **carrying capacity** 环境容纳量 L:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right).$$

> Growth is fastest at $P = \frac{L}{2}$, and $P \to L$ as $t \to \infty$.

Logistic growth (BC only), in detail



logistic: $dP/dt = kP(1 - P/L)$ · $P \to L$ as $t \to \infty$

> Real populations cannot grow for ever.

Exponential Models with Differential Equations

> A cooling cup of coffee: Newton's law of cooling is a classic differential equation model

Exam tips

- Solve a **separable** equation by getting all y on one side and all x on the other, then integrating both sides (add $+C$ once).
- Use the initial condition to find C (a particular solution).
- Sketch or read a **slope field**: the little segments show $\frac{dy}{dx}$ at each point, and a solution curve follows them.
- Recognise **exponential** models $\frac{dy}{dt} = ky \Rightarrow y = Ce^{kt}$ (growth/decay).
- A differential equation gives the slope – you must integrate to recover the function.

6 Recap

Unit 7 – key takeaways

1 Modelling
a sentence about a rate becomes an equation in dy/dt

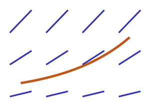
2 Slope fields
segments show the slope; a solution follows them

3 Separation
split the variables, integrate, add $+C$ once

4 Exponentials
$dy/dt = ky \Rightarrow y = y_0e^{kt}$ – growth or decay

Check yourself

1. Write "the rate of decay is proportional to the amount".
2. Solve $\frac{dy}{dx} = xy$ in general form.
3. How many times do you add $+C$ when separating?
4. Does half-life depend on the starting amount?



a solution follows the field

Answers **1.** $\frac{dy}{dt} = -ky$ **2.** $y = Ae^{x^2/2}$ **3.** once, on the x-side **4.** no