Integration & Accumulation of Change

AP Calculus BC ▪ Unit 6

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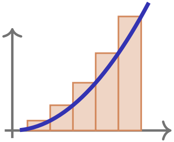
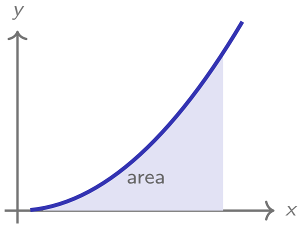


What we will cover

1. Accumulation of change
2. Riemann sums
3. The definite integral
4. The Fundamental Theorem
5. Antiderivatives
6. u-substitution
7. BC techniques



1 Accumulation

Accumulation of change

Differentiation found rates. Integration runs it in reverse: given a rate, find the **accumulated change** 累积变化.

> The area between a rate graph and the x-axis is the total accumulation.

Water tank

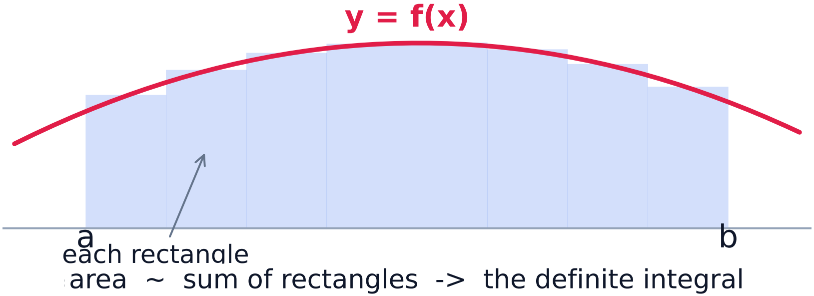
2 Riemann sums

Approximating area with a Riemann sum

Split the interval and add rectangle areas – a **Riemann sum** 黎曼和. Each rectangle is $f(x)\,\Delta x$.

> As the strips narrow, the sum approaches the definite integral.

Approximating area with a Riemann sum, in detail



$$\sum f(x_i^*)\Delta x \to \int_a^b f(x)\,dx \text{ as } n \to \infty$$

Split the interval and add rectangle areas – a **Riemann sum** 黎曼和.

The four standard estimates

Left sum – height from each strip's left endpoint

Right sum – height from the right endpoint

Midpoint sum – height from the **midpoint** 中点

Trapezoidal sum 梯形法 – average the two endpoint heights

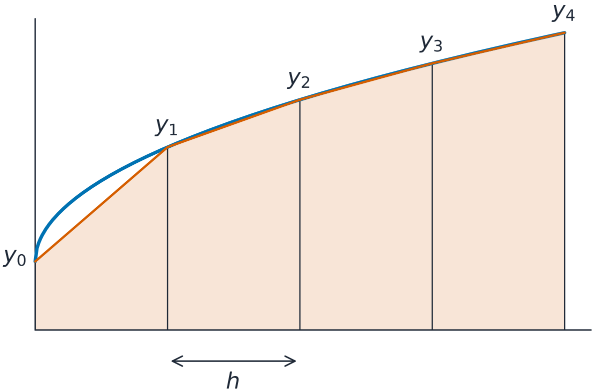
Strips may be uniform or nonuniform – read the widths from the table.

Over- or underestimate?

Judge from the behaviour of the function:

- increasing f: left sum under, right sum over
- concave up: trapezoidal over
- concave down: trapezoidal under

Over- or underestimate?, in detail



Exam parts ask you to state which and why.

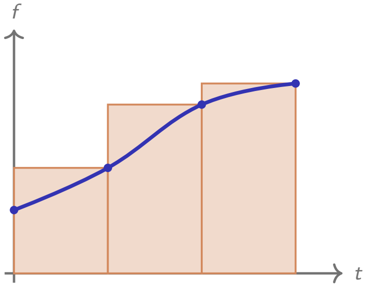
Worked example – a right sum from a table

$f(0) = 3,\ f(2) = 5,\ f(4) = 8,\ f(6) = 9$.
Estimate $\int_0^6 f$ with a right sum, $n = 3$.

$\Delta x = 2$; use each strip's right endpoint:

$$2\big(f(2) + f(4) + f(6)\big) = 2(5 + 8 + 9) = 44.$$

f is increasing, so this is an overestimate; the left sum $2(3 + 5 + 8) = 32$ would be an underestimate.



From Riemann sum to integral notation

As the widths shrink to zero the sum approaches an exact value – the **definite integral** 定积分:

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

A definite integral is the limit of a Riemann sum – be ready to translate each into the other.

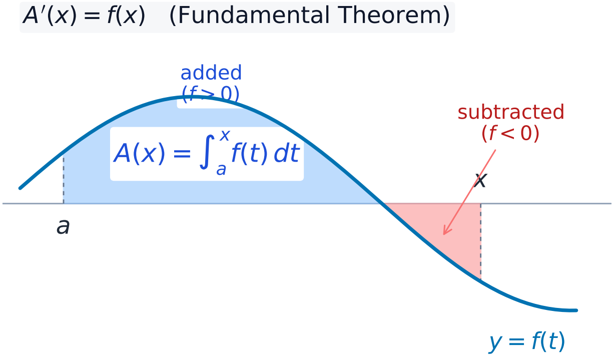
3 The Fundamental Theorem

Accumulation functions and the FTC

A definite integral with a variable upper limit is an **accumulation function** 累积函数. The **Fundamental Theorem** 微积分基本定理 says differentiation undoes it:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

> So $g(x) = \int_a^x f$ has $g' = f$ and $g'' = f'$.



Behaviour of an accumulation function

Because $g'(x) = f(x)$, everything from Unit 5 applies – read it off the graph of f:

g **increasing** where $f > 0$,
decreasing where $f < 0$

g has a **local extremum**
where f crosses zero

g **concave up** where f is increasing

a **value** of g is the
signed area up to x

Accumulation functions, the FTC, and improper integrals

since $g'(x) = f(x)$	the graph of f tells you everything about g: g increases where $f > 0$, decreases where $f < 0$, and turns where f crosses zero
concavity of g	follows the *slope* of f
the FTC 微积分基本定理	its evaluation form computes a definite integral from an **antiderivative** 原函数: $\int_a^b f = F(b) - F(a)$
a trapezoidal sum 梯形和	averages the left and right sums; it overestimates a concave-up function
an improper integral converges 收敛	if the limit exists: $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = 1$
it diverges 发散	if the limit does not – $\int_1^\infty \frac{1}{x}\,dx$ is the standard case

> Write the limit explicitly before evaluating an improper integral. Skipping straight to the antiderivative loses the mark even when the answer is right.

Properties of definite integrals

$$\int_a^a f = 0$$

$$\int_b^a f = -\int_a^b f$$

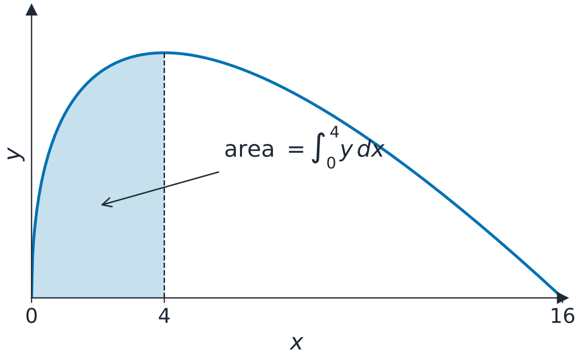
$$\int_a^b kf = k\int_a^b f$$

$$\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$$

$$\int_a^c f + \int_c^b f = \int_a^b f$$

> The last one – splitting at an interior point – lets you build a total from pieces read off a graph.

A definite integral is the signed area



A definite integral is the signed area between the curve and the x-axis

Evaluating an integral – the second part of the FTC

If $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Find any **antiderivative** 原函数, then subtract its values at the limits.

Net change: $\int_a^b g'(t)\,dt = g(b) - g(a)$, so $g(5) = g(0) + \int_0^5 g'$.

Evaluate $\int_1^3 (2x+1)\,dx$

An antiderivative is $F(x) = x^2 + x$:

$F(3) - F(1) = (9+3) - (1+1) = 10.$

4 Antiderivatives

Antiderivatives and indefinite integrals

An **indefinite integral** 不定积分 is the family of all antiderivatives:

$$\int f(x)\,dx = F(x) + C.$$

> Never drop the **constant of integration** 积分常数.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^x dx = e^x + C$$

$$\int \cos x\, dx = \sin x + C$$

u-substitution – reversing the chain rule

Choose an inside function $u = g(x)$, so $du = g'(x)\,dx$:

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du.$$

> Look for a function and its derivative both present.

Evaluate $\int 2x\cos(x^2)\,dx$

Inside is $u = x^2$, and $du = 2x\,dx$ is already there:

$$\int \cos u\,du = \sin u + C = \sin(x^2) + C.$$

Spotting that $2x$ is exactly $\frac{du}{dx}$ is the whole trick.

For a definite integral: change the limits to u-values, or convert back to x first.

Integrating Using Long Division and Completing the Square

When a rational integrand is "top-heavy" (numerator degree $\geq$ denominator degree), **long division** rewrites it as a polynomial plus a proper fraction you can integrate.

> **Completing the square** in a denominator turns it into a form like $u^2 + a^2$, leading to an **arctangent** antiderivative $\frac{1}{a}\arctan\frac{u}{a} + C$.

Integrating Using Integration by Parts

Integration by parts 分部积分 reverses the product rule:

$$\int u\,dv = uv - \int v\,du.$$

Choose u to simplify when differentiated a.

> It handles products like $\int xe^x\,dx$ and $\int x\ln x\,dx$, sometimes applied twice.

Integrating Using Linear Partial Fractions

Partial fractions 部分分式 split a rational function with a factorable denominator into a sum of simpler fractions:

$$\frac{1}{(x-a)(x-b)} = \frac{1}{a-b}\left(\frac{1}{x-a} - \frac{1}{x-b}\right).$$

This technique is essential for the logistic differential equation in the next unit.

Reshaping an integrand, and choosing a technique

Two algebraic set-up moves:

- **long division** 多项式长除法 when the top degree $\geq$ the bottom
- **completing the square** 配方法 to reach an arctangent or log form

try a **basic** antiderivative

↓

look for a u-**substitution**

↓

reshape with **algebra**

Naming the structure first prevents wasted effort.

5 Techniques

BC techniques for antidifferentiation

Integration by parts 分部积分
$\int u\,dv = uv - \int v\,du$

Partial fractions 部分分式
split a rational into simple pieces

Improper integrals 反常积分
a limit at ∞ or a discontinuity

Choose the technique by the form: a product $\rightarrow$ by parts, a rational $\rightarrow$ partial fractions, an infinite bound $\rightarrow$ an improper integral (evaluate with a limit).

Exam tips

- Integration is antidifferentiation; use the **power rule** $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ and don't forget the $+C$.
- The **Fundamental Theorem** links the two: $\int_a^b f'(x)\,dx = f(b) - f(a)$, and $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.
- Approximate a definite integral with **Riemann sums** or the trapezoidal rule from a table of values.
- A definite integral is a signed area (below the axis counts negative); split at sign changes for total area.
- Use **u-substitution** and remember to change the limits (or back-substitute) accordingly.

6 Recap

Unit 6 – key takeaways

1 Accumulation

signed area under a rate graph is the total change

2 Riemann sums

left, right, midpoint, trapezoid – and say over or under

3 The FTC

$\frac{d}{dx}\int_a^x f = f(x)$ and $\int_a^b f = F(b) - F(a)$

4 u-substitution

spot a function and its derivative; never drop $+C$

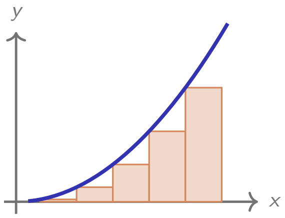
Check yourself

1. For an increasing f, is a left sum an over- or underestimate?

2. What is $\frac{d}{dx} \int_a^x f(t)\, dt$?

3. Evaluate $\int_1^3 (2x+1)\, dx$.

4. In $\int 2x\cos(x^2)\, dx$, what is u?



left sum

Answers **1.** underestimate **2.** $f(x)$ **3.** 10 **4.** $u = x^2$