Analytical Applications of Differentiation

AP Calculus BC ▪ Unit 5

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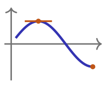
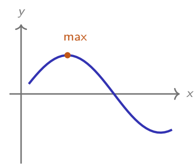


What we will cover



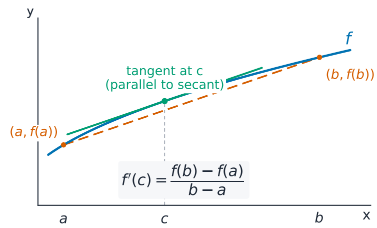
1 Mean Value Theorem

The Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then for some c in (a, b):

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Somewhere inside, the instantaneous rate equals the average rate – some tangent is parallel to the secant.



Exam skill – justifying the MVT

The **Mean Value Theorem** 中值定理 is an existence theorem, so the marks are in the justification:

1. state f is continuous on $[a, b]$ and differentiable on (a, b)
2. compute $\dfrac{f(b) - f(a)}{b - a}$
3. conclude "by the MVT there is a c in (a, b) with $f'(c)$ equal to that value"

Both hypotheses must be named – dropping one loses the mark even with the right c.

Verify the MVT for $f(x) = x^2$ on $[1, 3]$

A polynomial, so continuous and differentiable everywhere. Average rate $\frac{9-1}{2} = 4$; solve $f'(c) = 2c = 4$ to get $c = 2$, which lies in $(1, 3)$.

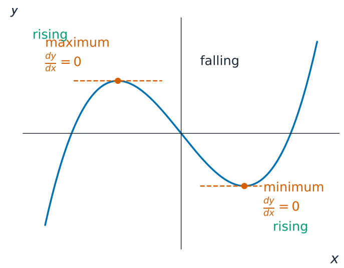
2 Critical points

Extreme values and critical points

The **Extreme Value Theorem** 极值定理: a function continuous on a closed interval attains both an absolute max and an absolute min. A **critical point** 临界点 is an interior point where $f'(x) = 0$ or f' does not exist.

Garden fence

At a maximum or minimum the derivative is zero



At a maximum or minimum the derivative is zero

Where a Function Increases or Decreases

The **first derivative** tells you where f rises or falls:

- $f'(x) > 0$ on an interval $\Rightarrow$ f is **increasing 递增** there;
- $f'(x) < 0$ $\Rightarrow$ f is **decreasing 递减**.

Increasing and decreasing intervals

The first derivative says where f rises or falls:

- $f'(x) > 0 \;\Rightarrow\;$ f is **increasing** 递增
- $f'(x) < 0 \;\Rightarrow\;$ f is **decreasing** 递减

Mountain extrema

The first derivative test

Classify a critical point c by how f' changes sign there:

f' turns $+ \to -$ **local maximum** 极大值	f' turns $- \to +$ **local minimum** 极小值	f' does **not** change sign **neither**

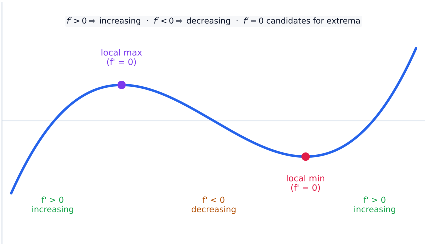
Always state the sign change as your justification.

Worked example – classify the critical points

$f(x) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x = 3x(x-2)$, zero at $x = 0,\ 2$. Signs: $f' > 0$ for $x < 0$; $f' < 0$ on $(0, 2)$; $f' > 0$ for $x > 2$. $+ \to -$ at $x = 0$: local max, $f(0) = 0$. $- \to +$ at $x = 2$: local min, $f(2) = -4$.

Worked example – classify the critical points, in detail



$f(x) = x^3 - 3x^2$ $f'(x) = 3x^2 - 6x = 3x(x-2)$, zero at $x = 0,\ 2$.

The candidates test – absolute extrema

On a closed interval, absolute extrema occur only at critical points or endpoints:

1. list every critical point in $[a, b]$ and both endpoints
2. evaluate f at each candidate
3. largest is the absolute max; smallest the absolute min

Show the table of values – the comparison *is* the argument.

endpoints a, b
$+$
every critical point
$=$ the full candidate list

Extrema: local, global, and how to justify

Local (relative) extrema	**Local (relative) extrema** are the highest or lowest value in some neighbourhood – and by Fermat's theorem they occur only at critical points.
Absolute extrema	On a closed interval these are found by the candidates test: evaluate at every critical point *and* both endpoints.
Justifying	The first-derivative test names the sign change of f'; the second-derivative test uses the sign of f'' at the critical point.
What earns the mark	State the value *and* the reason – "f' changes from positive to negative at $x = 2$, so $f(2)$ is a local maximum".

A critical point is a candidate, never a conclusion. The justification is the marked part of an AP answer.

3 Concavity

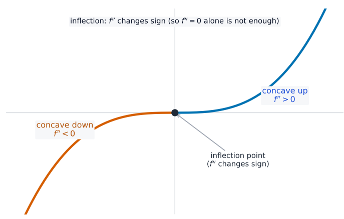
Concavity and points of inflection

The second derivative describes bending:

- $f'' > 0$: **concave up** 上凹 (f' increasing)
- $f'' < 0$: **concave down** 下凹 (f' decreasing)

A **point of inflection** 拐点 is where f'' changes sign – not merely where $f'' = 0$.

Concavity and points of inflection, in detail



Concavity comes from the sign of the second derivative; it flips at an inflection point

The second derivative test

An alternative at a critical point c with $f'(c) = 0$:

- $f''(c) > 0 \Rightarrow$ **local minimum**
- $f''(c) < 0 \Rightarrow$ **local maximum**
- $f''(c) = 0 \Rightarrow$ **inconclusive**

If $f''(c) = 0$, fall back on the first derivative test.

Special case: if a continuous function has only one critical point on an interval and it is a local extremum, it is the absolute extremum there.

Connecting f, f' and f''

A very common setup: you are given the graph of f' and asked about f.

- f **increasing** $\Leftrightarrow$ f' above the axis
- f has a **local max** $\Leftrightarrow$ f' crosses $+ \rightarrow -$
- f **concave up** $\Leftrightarrow$ f' increasing
- f has an **inflection** $\Leftrightarrow$ f' has a local extremum

Answer questions about f using the height and slope of the f' graph.

4 Optimization

Solving an optimization problem

Optimization 最优化 is the same machinery aimed at a real goal:

1. write the quantity as a function of one variable (use the constraint to eliminate the rest)
2. state the allowed interval
3. find and test the critical points
4. answer with units and interpretation

constraint $\rightarrow$ one variable

differentiate, set $= 0$

test, then interpret

Worked example – the farmer's fence

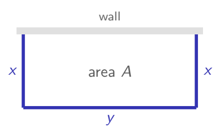
100 m of fence, three sides (a wall is the fourth). Maximize the area.

Ends x, side y: constraint $2x + y = 100$, so $y = 100 - 2x$.

$$A(x) = x(100 - 2x) = 100x - 2x^2$$

$$A'(x) = 100 - 4x = 0 \;\Rightarrow\; x = 25$$

$A'' = -4 < 0$, a maximum: $y = 50$ and $A = 1250\ \mathrm{m}^2$.



Behaviour of implicit relations

It all extends to implicitly defined relations:

- horizontal tangent where $\dfrac{dy}{dx} = 0$
- vertical tangent where $\dfrac{dy}{dx}$ is undefined

$\dfrac{dy}{dx}$ usually involves both x and y – substitute it back in when finding $\dfrac{d^2y}{dx^2}$, then read concavity from the sign.

5 Recap

Unit 5 – key takeaways

1 MVT

some tangent is parallel to the secant – name both hypotheses

2 Critical points

$f' = 0$ or undefined: candidates only, so test each

3 The two tests

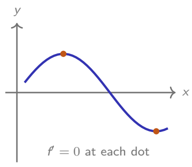
f' sign change, or $f'' > 0$ min / $f'' < 0$ max

4 Justify

cite the sign of f' or f'' – the reasoning earns the marks

Check yourself

1. f' turns $-$ to $+$ at c. What is c?
2. $f''(c) = 0$ at a critical point. What does the test tell you?
3. Where can an absolute max on $[a, b]$ occur?
4. Is every critical point an extremum?



$f' = 0$ at each dot

Answers **1.** a local minimum **2.** inconclusive – use the f' test **3.** a critical point or an endpoint **4.** no