Chain Rule, Implicit & Inverse

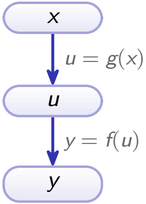
AP Calculus BC ▪ Unit 3

AP Calculus BC

$$\boxed{x} \rightarrow \boxed{g} \xrightarrow{u} \boxed{f} \rightarrow \boxed{y}$$

What we will cover

1. Composite functions
2. The chain rule
3. Implicit differentiation
4. Derivatives of inverse functions
5. Inverse trig derivatives
6. Higher-order derivatives & strategy



1 Chain rule

What makes a function composite?

A **composite function** 复合函数 $f(g(x))$ nests one function inside another.

> Always ask two questions: what is the inside, and what is its derivative?

Chain mountain

The chain rule

Differentiate outer-then-inner:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, with $y = f(u)$ and $u = g(x)$:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

> The inner derivative $g'(x)$ is the most-forgotten factor on the exam.

differentiate the **outer**, leave the in-

multiply by the derivative of the **inside**

$\cos(x^2) \cdot 2x$

Worked example – a power of a polynomial

Differentiate $h(x) = (2x^2+1)^5$

Outer is $(\,\cdot\,)^5$, inner is $2x^2+1$:

$$h'(x) = 5(2x^2+1)^4 \cdot \frac{d}{dx}(2x^2+1)$$

$$= 5(2x^2+1)^4 \cdot 4x = 20x(2x^2+1)^4.$$

outer $(\,\cdot\,)^5 \to 5(\,\cdot\,)^4$

$\downarrow$

inner $2x^2+1 \to 4x$

Exam skill – the chain rule from a table

A table gives values; no formula is needed. With $h(x) = f(g(x))$:

$$h'(a) = f'\big(g(a)\big) \cdot g'(a).$$

> Read $g(a)$ first, then look up f' at that value – not at a.

Find $h'(1)$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	3	7	1	9

$g(1) = 2$, so
$h'(1) = f'(2) \cdot g'(1) = 7 \times 5 = 35$.

2

Implicit

Implicit differentiation

Some curves are defined **implicitly** 隐式地 – an equation in x and y not solved for y, e.g. $x^2 + y^2 = 25$.

1. differentiate both sides in x
2. each y-term picks up a $\frac{dy}{dx}$ (chain rule)
3. solve algebraically for $\frac{dy}{dx}$

$$x^2 + y^2 = 25$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Worked example – tangent to a circle

Tangent to $x^2 + y^2 = 25$ at $(3, 4)$

From $\dfrac{dy}{dx} = -\dfrac{x}{y}$, at $(3, 4)$:

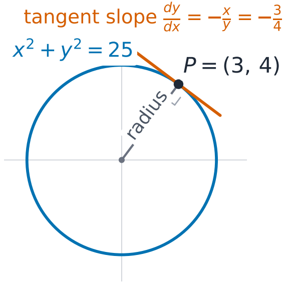
$$\frac{dy}{dx} = -\frac{3}{4}.$$

So the tangent is

$$y - 4 = -\tfrac{3}{4}(x - 3).$$

The tangent is perpendicular to the radius – a free sanity check.

Worked example – tangent to a circle, in detail



$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$ · tangent ⊥ radius

> Implicit differentiation of $x^2 + y^2 = 25$ gives $\dfrac{dy}{dx} = -\dfrac{x}{y}$, so at $(3, 4)$ the gradient is $-\frac{3}{4}$ – perpendicular to the radius.

Exam skill – "Show that $\frac{dy}{dx} = \dots$"

The target is given, so the marks are in the algebra, not the answer:

1. differentiate both sides
2. product/chain rule on mixed xy terms
3. collect every $\frac{dy}{dx}$ on one side
4. factor, then divide

> A correct final line that skips the algebra earns little.

Horizontal tangent

$\frac{dy}{dx} = 0$: numerator $= 0$

Vertical tangent

denominator $= 0$

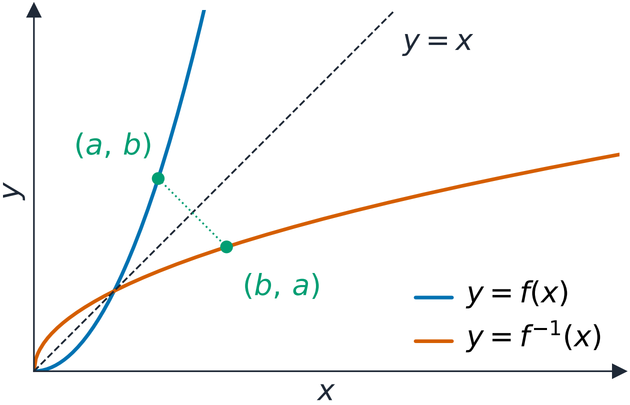
3 Inverse functions

Derivatives of inverse functions

If g is the **inverse function** 反函数 of f (so $f(g(x)) = x$):

$$g'(x) = \frac{1}{f'\left(g(x)\right)}.$$

> Slopes at matching points are **reciprocals** 倒数 – the graphs are mirror images in $y = x$.



Worked example – inverse from a point

$f(2) = 5$ and $f'(2) = 3$; g is the inverse of f

$(2, 5)$ lies on f, so $(5, 2)$ lies on g. Then

$$g'(5) = \frac{1}{f'(2)} = \frac{1}{3}.$$

> Swap the coordinates first, then take the reciprocal of f' at the original x.

$$\boxed{\text{on } f\text{: } (2,5)} \xrightarrow{\text{swap}} \boxed{\text{on } g\text{: } (5,2)}$$

Inverse trigonometric derivatives

The same idea gives the **inverse trigonometric functions** 反三角函数 you must know:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$$

Chain them when the input is a function: $\dfrac{d}{dx}\arctan(3x) = \dfrac{3}{1+9x^2}$.

4 Higher-order & strategy

Higher-order derivatives

Differentiating f' gives the **second derivative** 二阶导数 f'':

$$f''(x) = \frac{d^2y}{dx^2} = y'', \qquad f^{(n)}(x) = \frac{d^ny}{dx^n}.$$

> f'' measures how the slope is changing – it drives **concavity** 凹凸性 (Unit 5) and acceleration (Unit 4).

$s(t)$ position

↓

$s'(t)$ velocity

↓

$s''(t)$ acceleration

Choosing the right rule

Read the structure from the outside in: name the outermost operation, apply its rule, recurse inward.

a **sum**? differentiate term by term	a **product** / **quotient**? that rule (chain inside)
a **composite**? chain rule	given **implicitly**? implicit differentiation

> Neatness prevents the sign and bookkeeping errors that cost the marks.

5 Recap

Unit 3 – key takeaways

1 **Chain rule**

$f'(g(x)) \cdot g'(x)$ – never drop the inner derivative

2 **Implicit**

differentiate both sides; each y gives a dy/dx; then solve

3 **Inverse**

$g'(x) = 1/f'(g(x))$ – reciprocal slopes at swapped points

4 **Second derivative**

differentiate twice; velocity $\to$ acceleration

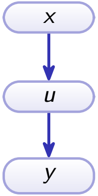
Check yourself

1. Differentiate $\cos(3x)$.

2. For $x^2 + y^2 = 25$, what is $\frac{dy}{dx}$ at $(3, 4)$?

3. $f(2) = 5$, $f'(2) = 3$. If $g = f^{-1}$, what is $g'(5)$?

4. Which factor do students most often forget?



$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Answers **1.** $-3\sin(3x)$ **2.** $-\frac{3}{4}$ **3.** $\frac{1}{3}$ **4.** the inner derivative $g'(x)$