Limits & Continuity

AP Calculus BC ▪ Unit 1

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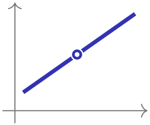


What we will cover

1 Change at an instant

Can change occur at an instant?

A speedometer reads 60 km/h. But at a single instant no time passes and no distance is covered:

$$\text{speed} = \frac{0}{0} \;\text{– undefined.}$$

The clever move: do not plug in zero. Watch what the average rate **approaches** 趋近 as the interval shrinks.

That approaching value is a **limit** 极限 – it powers the derivative (Unit 2) and, in reverse, the integral (Unit 6).

What calculus is for, and the words limits are described in

calculus	the mathematics of **change** 变化 and of **accumulation** 累积
the two questions	how fast is something changing at an instant, and how much accumulates over an interval
a limit	the value $f(x)$ approaches as x approaches a – and it ignores the point itself
the graph's distinction	a hole means the limit exists and the value does not; a filled dot elsewhere means they differ
average rate of change 平均变化率	the slope of a secant; the derivative is the limit of it
not tested	the epsilon–delta definition is not on the AP exam, so it is not used here

> "Ignore the point itself" is the whole idea. A limit describes the approach, and a function can be undefined exactly where its limit is clearest.

2 Reading a limit

Defining a limit

$\lim_{x \to c} f(x) = R$ means $f(x)$ can be made **arbitrarily** 任意地 close to R by taking x **sufficiently** 足够 close to c – but not equal to c.

> A limit describes the behaviour near c, not the value at c.

graphically 用图象

numerically 用数值

analytically 用解析式

Moving between the three is a core skill.

Reading a limit from a graph

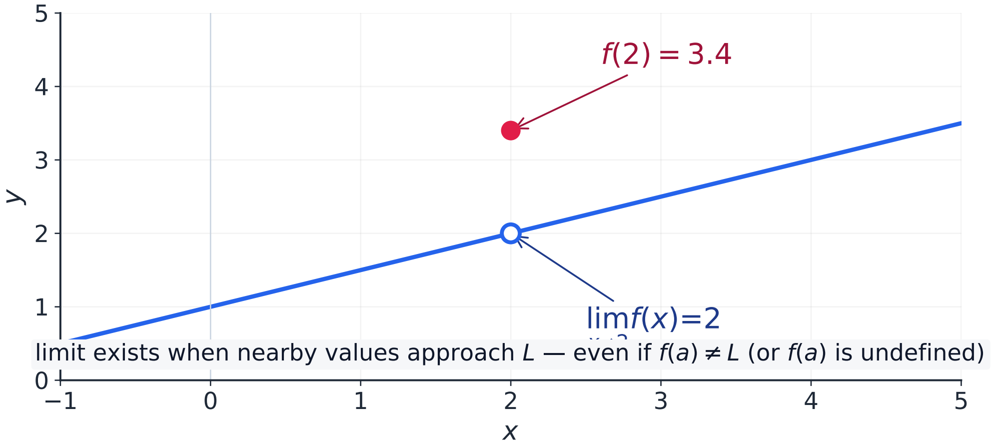
Trace toward c from each side:

- from the left: **left-hand limit** 左极限
- from the right: **right-hand limit** 右极限

If both head to the same height, the two-sided limit exists.

> **Open circle** 空心圆: a height approached but not reached. **Closed circle** 实心圆: the actual value.

Reading a limit from a graph, in detail



The open circle is the height the curve approaches (the limit); the filled dot is the actual value $f(2)$ – they need not agree.

When a limit does not exist

Jump 跳跃

the two sides disagree

$$\lim_{x\to 0} \frac{|x|}{x} \text{ DNE}$$

Unbounded 无界

grows without limit

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

Oscillates 振荡

forever near c

$$\lim_{x\to 0} \sin\frac{1}{x} \text{ DNE}$$

> Watch the **scale** 比例: a zoomed-out graph can hide behaviour near a point – confirm with algebra.

Estimating a limit from a table

Estimate $\lim_{x\to 2} \frac{x^2-4}{x-2}$ (it is $\frac{0}{0}$ at $x=2$)

x	1.9	1.99	1.999	$\to 2 \leftarrow$	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	?	4.001	4.01	4.1

Both sides march toward 4.

A table only suggests a value – it is a numerical estimate, not a proof.

3 Computing limits

The limit theorems

If both limits exist, the limit of a combination is that combination of the limits:

$$\lim(f \pm g) = \lim f \pm \lim g$$

$$\lim(fg) = \lim f \cdot \lim g$$

$$\lim \frac{f}{g} = \frac{\lim f}{\lim g} \quad \text{(bottom} \neq 0)$$

$$\lim g(f(x)) = g(\lim f(x)) \quad (g \text{ continuous})$$

The practical rule: try **direct substitution** 直接代入 first. A real number out means you are done.

The vocabulary a limit question uses

Piecewise-defined function	A **piecewise-defined function** uses different rules on different intervals – so the one-sided limits at a join must be checked separately.
Composite	For a **composite** $f(g(x))$, the limit passes inside when f is continuous there: $\lim f(g(x)) = f(\lim g(x))$.
Radical	A **radical** expression with an indeterminate form usually needs the conjugate – multiply top and bottom by it and the difference of squares clears the root.
Continuous on an interval	A function is **continuous on an interval** when it is continuous at every point of it – and one-sided at the endpoints.
Existence theorem	An **existence theorem** – the Intermediate Value Theorem, the Extreme Value Theorem – guarantees that something exists without saying where.

> The **instantaneous rate of change** is the limit of an average rate as the interval shrinks. That sentence is the whole idea of the derivative.

When substitution gives $\frac{0}{0}$

$\frac{0}{0}$ is an **indeterminate form** 未定式 – it does not mean the limit fails. Rewrite into an **equivalent form** 等价形式, then substitute.

> The cancelled factor is why the graph had a hole: the two functions agree everywhere except at c.

Three standard moves

Factor and cancel 因式分解并约分:

$$\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4$$

Multiply by the **conjugate** 共轭:

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} = \frac{1}{2}$$

Or use a trig identity.

Choosing a procedure

1. substitute – a real answer means done

2. $\frac{0}{0}$? rewrite, then substitute

3. $\frac{5}{0}$? infinite or DNE – check each side

4. $x \to \pm\infty$? compare the **dominant** 主导 terms

5. trapped between two functions? squeeze theorem

Choosing a method for a limit

1. substitute	if it gives a number, that is the limit – for a continuous function
2. factor and cancel	for $0/0$ from a common factor
3. rationalise	multiply by the conjugate when a surd causes the $0/0$
4. use identities	alternate forms of trigonometric functions often simplify what algebra cannot
5. squeeze	trap the function between two with the same limit
6. compare degrees	for a limit at infinity, the leading terms decide

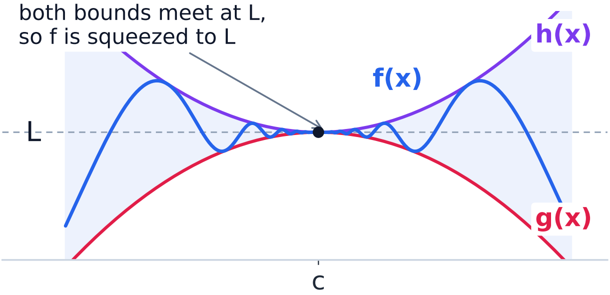
This is a skill topic, not new content: choose the right tool for the limit in front of you. And the same limit lives in a graph, a table and an algebraic form – move between them.

The squeeze theorem

If $g(x) \leq f(x) \leq h(x)$ near c and both bounds meet at L, then f has nowhere to go:

$$\lim_{x \to c} f(x) = L.$$

> The two famous results proved this way: $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$

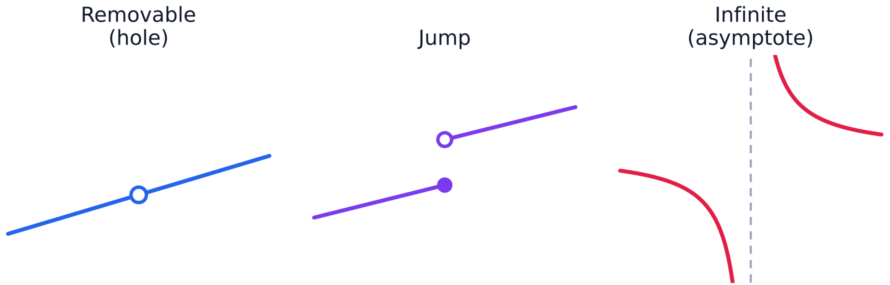


if $g \leq f \leq h$ near c and $\lim g = \lim h = L$, then $\lim f = L$

4 Continuity

Three types of discontinuity

A function is **discontinuous** 间断 at c when its graph breaks there:



f continuous at a iff $\lim_{x\to a} f(x) = f(a)$ (limit exists and equals the value)

Removable 可去间断 (a hole) ▪ **jump** 跳跃间断 (sides disagree) ▪ **infinite** 无穷间断 (an asymptote).

Continuity at a point – the three-part test

f is **continuous** 连续 at $x = c$ exactly when all three hold:

$f(c)$ **exists**

$\lim_{x \to c} f(x)$ **exists**

they **agree**: $\lim_{x \to c} f(x) = f(c)$

> The point is there, the limit is there, and the two agree. If any one fails, f is discontinuous.

Learn it as a checklist – it is the backbone of nearly every continuity question.

Continuity over an interval, and repairing a hole

You rarely check point by point – whole families are continuous on their domains:

> Polynomial, rational, power, **exponential** 指数, **logarithmic** 对数 and **trigonometric** 三角 functions are continuous at every point of their domains.

So a rational function is continuous everywhere except where its denominator is zero.

Removing a discontinuity

If the limit exists at a hole, redefine the function there to equal the limit – the graph is repaired.

For a **piecewise** 分段函数 function at a boundary, solve for the **parameter** 参数 that makes the pieces meet:

$$\lim_{x \to c^-} f = \lim_{x \to c^+} f = f(c).$$

5 Asymptotes

Infinite limits and vertical asymptotes

Where a non-zero number is divided by something approaching 0, the graph hugs the line $x = c$ – a **vertical asymptote** 垂直渐近线.

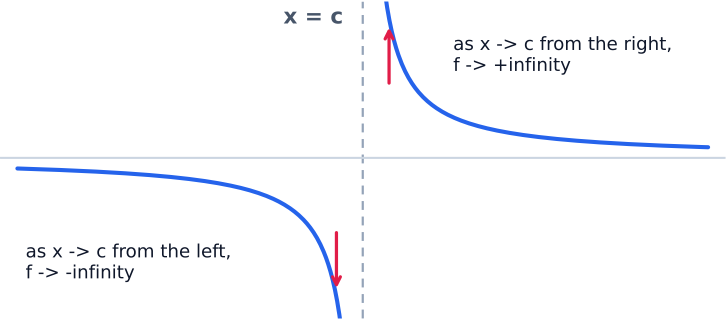
> Always check each side: the two sides can shoot opposite ways.

End behaviour, from the leading terms

Relative magnitudes	At infinity only the **relative magnitudes** matter: exponentials beat any polynomial, and a polynomial beats any logarithm.
Equal degrees	When numerator and denominator have the same degree, the limit is the **ratio of the leading coefficients** – and that ratio is the horizontal asymptote.
Unequal degrees	Numerator smaller: the limit is 0. Numerator larger: it grows without bound, and polynomial division gives the slant asymptote.
Naming it	That long-run **behavior** – the **end behavior** – is what a limit at infinity reports.

> Divide every term by the highest power in the denominator. Every rational end-behaviour limit falls out of that one step.

Infinite limits and vertical asymptotes, in detail



vertical asymptote $x = c$ when $\lim_{x \to c^+} f(x) = \pm\infty$ or $\lim_{x \to c^-} f(x) = \pm\infty$

Where a non-zero number is divided by something approaching 0, the graph hugs the line $x = c$ – a **vertical asymptote** 垂直渐近线.

Exploring Types of Discontinuities

A function is **discontinuous 间断** at c when its graph “breaks” there.

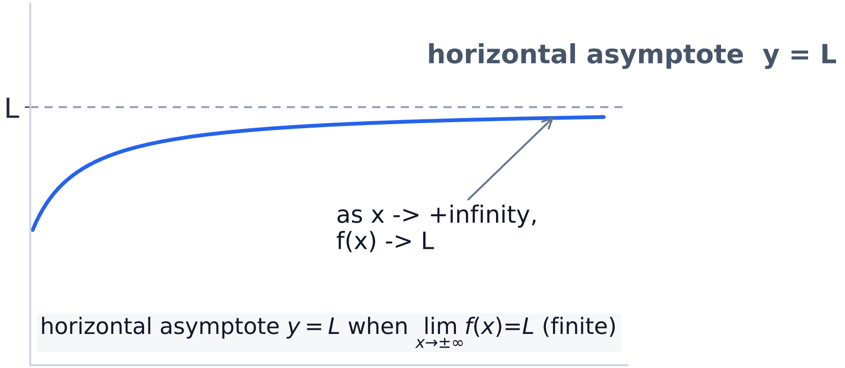
- **Removable discontinuity 可去间断** – a single hole. The two-sided limit exists, but the point is missing or misplaced.
- **Jump discontinuity 跳跃间断** – the two one-sided limits exist but disagree, so the curve jumps.
- **Infinite discontinuity 无穷间断** – the function blows up to $\pm\infty$ at a **vertical asymptote 垂直渐近线**.

Limits at infinity and horizontal asymptotes

Let the input grow: **end behaviour** 末端行为 as $x \to \pm\infty$. If outputs settle toward L, then $y = L$ is a **horizontal asymptote** 水平渐近线.

> Far out, an exponential beats any polynomial, and a polynomial beats any logarithm.

Limits at infinity and horizontal asymptotes, in detail



Let the input grow and watch the **end behaviour** 末端行为: if the outputs settle towards L, then $y = L$ is a **horizontal asymptote** 水平渐近线.

Rational functions – compare the degrees

top $<$ bottom limit is 0 (asymptote $y = 0$)	top $=$ bottom the **ratio of leading coefficients** 首项系数之比	top $>$ bottom unbounded – no horizontal asymptote

"As $t \to \infty$, where does the rate settle?" is answered with a limit at infinity.

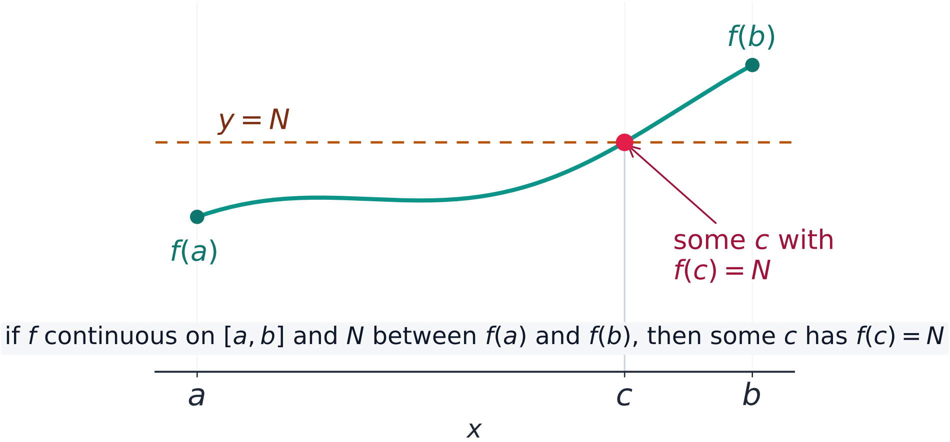
6 IVT

The Intermediate Value Theorem

If f is continuous on $[a, b]$ and d lies between $f(a)$ and $f(b)$, then $f(c) = d$ for at least one c in (a, b).

> An unbroken curve cannot skip a height between its endpoints.

The Intermediate Value Theorem, in detail

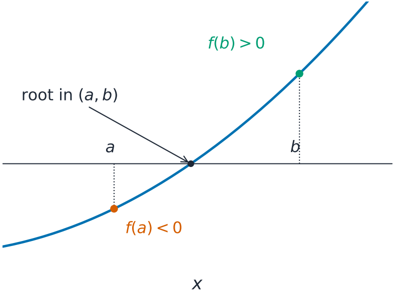


> If f is continuous on $[a, b]$ and d lies between $f(a)$ and $f(b)$, then $f(c) = d$ for at least one c in (a, b).

Exam skill – justifying with the IVT

These appear almost every year. A full-credit justification has three moves:

1. state continuity on $[a, b]$
2. show d is trapped: $f(a) < d < f(b)$
3. conclude by name: "by the IVT there is a c in (a, b) with $f(c) = d$"



Skipping the continuity statement – or not showing d is between the endpoints – loses the point.

Worked example – a hole at $x = 3$

Evaluate $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$

Direct substitution gives $\frac{0}{0}$, so factor:

$$\frac{(x-3)(x+3)}{x-3} = x + 3 \quad (x \neq 3)$$

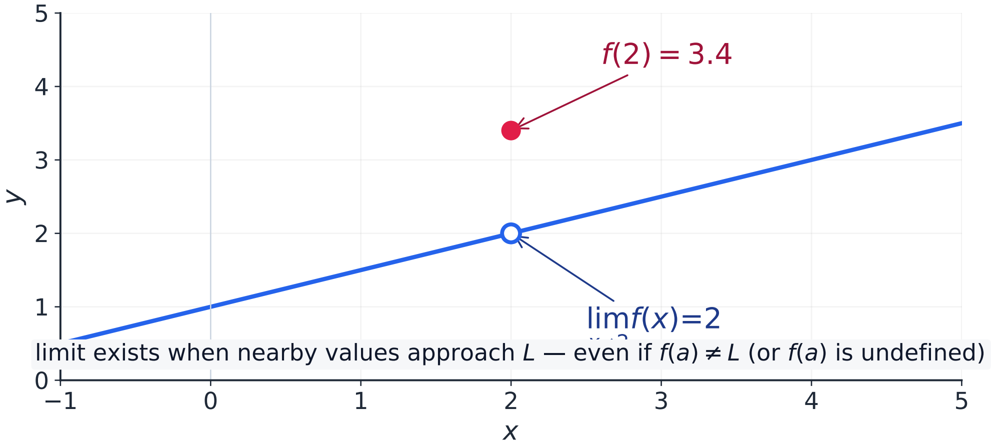
$$\lim_{x \to 3}(x+3) = 6.$$

The graph has a removable discontinuity at $x = 3$ – the limit exists even though the function is undefined there.

Defining Limits and Using Limit Notation

A smooth bridge curve: limits describe the value a graph approaches as we zoom in

Check yourself, in detail



> The open circle is the height the curve approaches (the limit); the filled dot is the actual value $f(2)$ – they need not agree.

Exam tips

- A limit describes what $f(x)$ **approaches**, which need not equal $f(a)$ – the two-sided limit exists only if both sides agree.
- Try direct substitution first; for a $\frac{0}{0}$ form, **factor and cancel** or rationalise before substituting.
- A function is **continuous** at a when the limit exists, $f(a)$ is defined, and they are equal.
- Use the **Intermediate Value Theorem** to guarantee a root: a continuous function that changes sign on $[a, b]$ takes every value between.
- Read horizontal asymptotes from end behaviour (limits at $\pm\infty$) and vertical asymptotes where the denominator (not the numerator) is zero.

7 Recap

Unit 1 – key takeaways

1 **A limit**

what f approaches near c – not the value at c

2 **Computing**

substitute first; on $0/0$ factor, cancel or rationalise

3 **Continuity**

value exists, limit exists, and the two agree

4 **The IVT**

continuous $+$ d trapped $\Rightarrow$ some c hits d

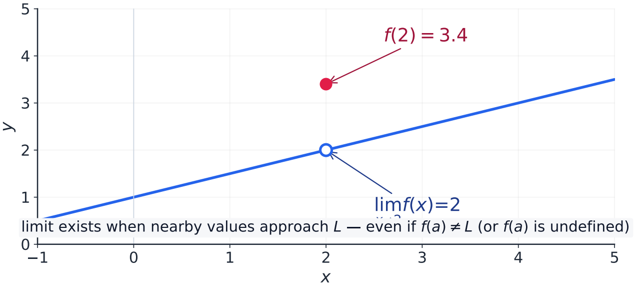
Check yourself

1. Evaluate $\lim_{x\to 3} \frac{x^2-9}{x-3}$.

2. Must $\lim_{x\to c} f(x)$ equal $f(c)$?

3. Name the three parts of the con test.

4. Which two conditions must an I' fication state?



Answers **1.** 6 **2.** no – only if f is continuous at c **3.** $f(c)$ exists, the limit exists, they agree **4.** continuity, and d between $f(a)$ and $f(b)$