AP CALCULUS BC • EXERCISE SHEET

9.8 Finding the Area of a Polar Region

Name: ______________ Class: __________ Date: ____________ **Total: 13 marks**

Objective

Build the skills to answer exam questions on the **area of a polar region**.

You must be able to:

- apply $A = \dfrac{1}{2}\displaystyle\int_{\alpha}^{\beta} \big(f(\theta)\big)^2 \, d\theta$
- choose the correct θ-limits that trace the region once
- set up the area for a full curve or a sector

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The polar area formula

The area swept by $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \big(f(\theta)\big)^2 \, d\theta.$$

The region is a fan of thin triangular sectors of area $\frac{1}{2}r^2 \, d\theta$.

A worked area

The circle $r = 2$: $A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} 2^2 \, d\theta = \dfrac{1}{2}(4)(2\pi) = 4\pi$ — matching πr^2.

Choosing the limits

For $r = 2\cos\theta$ (a circle traced once as θ goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$), integrate over exactly that range so the curve is traced **once**.

One petal of a rose

For $r = \sin(2\theta)$, one petal is traced as θ goes from 0 to $\frac{\pi}{2}$. Its area is $\dfrac{1}{2}\displaystyle\int_0^{\pi/2} \sin^2(2\theta) \, d\theta$.

2 Practice

Now apply the methods above.

2.1 State the polar area formula. [1]

2.2 Set up the area enclosed by $r = 3$ (a full circle). [2]

2.3 Evaluate $\frac{1}{2}\int_0^{2\pi} 9\,d\theta$. [2]

3 Exam-style questions

3.1 The area of a polar region is [1]

- **A** $\int_{\alpha}^{\beta} f(\theta)\, d\theta$
- **B** $\frac{1}{2}\int_{\alpha}^{\beta} \left(f(\theta)\right)^2 d\theta$
- **C** $\pi \int_{\alpha}^{\beta} f(\theta)\, d\theta$
- **D** $\frac{1}{2}\int_{\alpha}^{\beta} f(\theta)\, d\theta$

3.2 Consider $r = 1 + \cos\theta$ (a cardioid), traced once for $0 \le \theta \le 2\pi$.

(a) Set up the area integral. [2]

(b) Expand $(1 + \cos\theta)^2$ ready to integrate. [2]

3.3 One petal of the rose $r = 2\sin(3\theta)$ is traced as θ goes from 0 to $\frac{\pi}{3}$. Set up the area integral for one petal. [3]

Solutions

2.1 $A = \dfrac{1}{2}\displaystyle\int_{\alpha}^{\beta} \big(f(\theta)\big)^2 \, d\theta.$

2.2 $A = \dfrac{1}{2}\displaystyle\int_{0}^{2\pi} 3^2 \, d\theta.$

2.3 $\dfrac{1}{2}(9)(2\pi) = 9\pi.$

3.1 **B** — half the integral of r^2.

3.2 (a) $A = \dfrac{1}{2}\displaystyle\int_{0}^{2\pi} (1+\cos\theta)^2 \, d\theta.$ (b) $(1+\cos\theta)^2 = 1 + 2\cos\theta + \cos^2\theta.$

3.3 $A = \dfrac{1}{2}\displaystyle\int_{0}^{\pi/3} \big(2\sin(3\theta)\big)^2 \, d\theta = \dfrac{1}{2}\displaystyle\int_{0}^{\pi/3} 4\sin^2(3\theta) \, d\theta.$