

9.2 Second Derivatives of Parametric Equations

Name: _____ Class: _____ Date: _____

Total: 13 marks

KEY WORDS concavity 凹凸性

Objective

Build the skills to answer exam questions on the **second derivative of a parametric curve** — testing concavity.

You must be able to:

- compute $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$
- avoid the wrong “divide the second derivatives” shortcut
- use the sign of $\frac{d^2y}{dx^2}$ to state **concavity** 凹凸性

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The correct formula

The second derivative is **not** $\frac{d^2y/dt^2}{d^2x/dt^2}$. Instead, differentiate the first derivative with respect to t , then divide by $\frac{dx}{dt}$ again:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}.$$

■ A worked example

For $x = t^2$, $y = t^3$: $\frac{dy}{dx} = \frac{3t}{2}$. Then $\frac{d}{dt}\left(\frac{3t}{2}\right) = \frac{3}{2}$, and $\frac{dx}{dt} = 2t$, so

$$\frac{d^2y}{dx^2} = \frac{3/2}{2t} = \frac{3}{4t}.$$

■ Concavity

At $t = 1$, $\frac{d^2y}{dx^2} = \frac{3}{4} > 0$: the curve is **concave up** there. A negative value means concave down.

■ The two-step routine

1. Find $\frac{dy}{dx}$ (a function of t).
2. Differentiate it w.r.t. t and divide by $\frac{dx}{dt}$.

2 Practice

Now apply the methods above.

2.1 State the formula for $\frac{d^2y}{dx^2}$ for a parametric curve. [1]

2.2 For $x = t^2$, $y = t^3$, given $\frac{dy}{dx} = \frac{3t}{2}$, find $\frac{d}{dt}\left(\frac{dy}{dx}\right)$. [1]

2.3 Complete: find $\frac{d^2y}{dx^2}$ for that curve. [2]

3 Exam-style questions

3.1 The second derivative of a parametric curve is [1]

- A $\frac{d^2y/dt^2}{d^2x/dt^2}$
 - B $\frac{\frac{d}{dt}(dy/dx)}{dx/dt}$
 - C $\frac{dy/dt}{dx/dt}$
 - D $\frac{d}{dt}\left(\frac{dy}{dx}\right)$
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3.2 A curve has $x = t$, $y = t^3 - 3t$.

(a) Find $\frac{dy}{dx}$. [1]

(b) Find $\frac{d^2y}{dx^2}$. [2]

(c) State the concavity at $t = 1$. [1]

3.3 For $x = t^2$, $y = t^2 + t$, find $\frac{d^2y}{dx^2}$ at $t = 1$. [4]

Solutions

$$\mathbf{2.1} \quad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{dx/dt}.$$

$$\mathbf{2.2} \quad \frac{d}{dt}\left(\frac{3t}{2}\right) = \frac{3}{2}.$$

$$\mathbf{2.3} \quad \frac{d^2y}{dx^2} = \frac{3/2}{2t} = \frac{3}{4t}.$$

3.1 B — differentiate dy/dx w.r.t. t , then divide by dx/dt .

3.2 (a) $\frac{dy}{dx} = \frac{3t^2 - 3}{1} = 3t^2 - 3$. (b) $\frac{d}{dt}(3t^2 - 3) = 6t$; divide by $dx/dt = 1$: $\frac{d^2y}{dx^2} = 6t$. (c) at $t = 1$, $6 > 0$: concave up.

3.3 $\frac{dy}{dx} = \frac{2t+1}{2t}$; $\frac{d}{dt}\left(\frac{2t+1}{2t}\right) = \frac{d}{dt}\left(1 + \frac{1}{2t}\right) = -\frac{1}{2t^2}$; divide by $2t$: $\frac{d^2y}{dx^2} = -\frac{1}{4t^3}$; at $t = 1$: $-\frac{1}{4}$.