AP CALCULUS BC • EXERCISE SHEET

8.6 Finding the Area Between Curves That Intersect More Than Twice

Name: ______________ Class: __________ Date: ____________ **Total: 14 marks**

Objective

Build the skills to answer exam questions on **area when curves cross more than twice** — splitting the region where top and bottom switch.

You must be able to:

- find **all** intersection points
- decide which curve is on top on **each** subinterval
- add the pieces (each set up as top minus bottom)

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ When top and bottom swap

If two curves cross more than twice, the upper curve **changes** between crossings. You must split the integral at each crossing and use the correct top minus bottom on each piece.

■ Finding all crossings

For $y = x^3$ and $y = x$: $x^3 = x \Rightarrow x(x^2 - 1) = 0 \Rightarrow x = -1, 0, 1$. Three crossings, so **two** subintervals: $[-1, 0]$ and $[0, 1]$.

■ Deciding top on each piece

On $(0, 1)$ test $x = 0.5$: $x = 0.5$ vs $x^3 = 0.125$, so $y = x$ is on top. On $(-1, 0)$ test $x = -0.5$: $x^3 = -0.125$ vs $x = -0.5$, so $y = x^3$ is on top.

■ Adding the pieces

$$\text{Area} = \int_{-1}^{0} (x^3 - x)\, dx + \int_{0}^{1} (x - x^3)\, dx = \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.$$

Using $\int |f - g|$ automatically handles the switch, but you must still split to integrate.

2 Practice

Now apply the methods above.

2.1 Find all solutions of $x^3 = 4x$. [2]

2.2 On $(0, 2)$, which is greater: $y = 4x$ or $y = x^3$? (test $x = 1$) [1]

2.3 Evaluate $\int_0^2 (4x - x^3)\,dx$. [2]

3 Exam-style questions

3.1 When two curves cross three times, the total enclosed area needs [1]

- **A** one integral
- **B** the integral split into two pieces with the correct top each time
- **C** the average of the curves
- **D** integration in y only

3.2 Find the total area enclosed between $y = x^3$ and $y = 4x$.

(a) Find all intersection points. [2]

(b) By symmetry or otherwise, evaluate the total enclosed area. [4]

3.3 The curves $y = x^3 - x$ and $y = 0$ enclose two regions. Explain why $\int_{-1}^{1} (x^3 - x)\, dx = 0$ does **not** give the total area, and describe the correct setup. [2]

Solutions

2.1 $x^3 - 4x = x(x^2-4) = 0 \Rightarrow x = 0, \pm 2$.

2.2 $4x = 4$ vs $x^3 = 1$ at $x = 1$, so $y = 4x$ is greater.

2.3 $\left[2x^2 - \frac{x^4}{4}\right]_0^2 = 8 - 4 = 4$.

3.1 **B** — split at the crossings, using the correct upper curve on each piece.

3.2 (a) $x^3 = 4x \Rightarrow x = -2, 0, 2$. (b) On $(0,2)$ top is $4x$, area $\int_0^2 (4x - x^3)\,dx = 4$; by symmetry the region on $(-2,0)$ has equal area 4; total $= 8$.

3.3 The integral is 0 because the region below the axis (negative signed area) cancels the region above; the true area is $\displaystyle\int_{-1}^{0} (x^3 - x)\,dx + \left|\int_0^1 (x^3 - x)\,dx\right|$, i.e. split at $x = 0$ and add the absolute areas.