7.9 Logistic Models with Differential Equations

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS carrying capacity 环境容纳量 • logistic model 逻辑斯蒂模型

Objective

Build the skills to answer exam questions on the **logistic model** — growth limited by a carrying capacity.

You must be able to:

- read the **carrying capacity 环境容纳量** L from $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{L}\right)$
- state the long-run limit $\lim\limits_{t\to\infty} P = L$
- find where growth is **fastest** $(P = \frac{L}{2})$

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The logistic equation

Real growth slows as resources run out. The logistic model is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right),$$

where L is the **carrying capacity** — the population level the system can sustain.

■ Reading L off the equation

L is the value that makes the bracket zero. From $\dfrac{dP}{dt} = 0.03P\left(1 - \dfrac{P}{800}\right)$, the carrying capacity is $L = 800$.

■ The long-run limit

Whatever the (positive) start, P approaches L: $\lim\limits_{t\to\infty} P = L$. Here $P \to 800$. If P starts above L it decreases toward L; below L it increases toward L.

■ **Fastest growth at L/2**

The growth rate $\frac{dP}{dt}$ is largest at the **inflection point** $P = \frac{L}{2}$ — the middle of the S-shaped curve. Here that is $P = 400$.

2 Practice

Now apply the methods above.

2.1 For $\frac{dP}{dt} = 0.5P\left(1 - \frac{P}{2000}\right)$, state the carrying capacity. [1]

2.2 For the same model, state $\lim_{t\to\infty} P$ (given $P(0) = 100$). [1]

2.3 At what population is growth fastest for that model? [1]

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3 Exam-style questions

3.1 In $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{L}\right)$, growth is fastest when $P =$ [1]

- **A** 0
- **B** $\frac{L}{4}$
- **C** $\frac{L}{2}$
- **D** L

3.2 A fish population follows $\dfrac{dP}{dt} = 0.2P\left(1 - \dfrac{P}{5000}\right)$, with $P(0) = 500$.

(a) State the carrying capacity. [1]

(b) State $\lim_{t\to\infty} P$. [1]

(c) Find the population at which the population grows fastest. [1]

3.3 A logistic model has carrying capacity 1200 and $P(0) = 1500$. State, with a reason, whether the population increases or decreases toward the carrying capacity. [2]

Solutions

2.1 $L = 2000$.

2.2 2000.

2.3 $P = 1000$ (i.e. $L/2$).

3.1 **C** — fastest growth at the inflection point $P = \frac{L}{2}$.

3.2 (a) 5000. (b) 5000. (c) 2500.

3.3 Decreases — $P(0) = 1500 > 1200 = L$, so $1 - \frac{P}{L} < 0$ and $\dfrac{dP}{dt} < 0$, pulling P down toward 1200.