

6.2 Approximating Areas with Riemann Sums

Name: _____ Class: _____ Date: _____

Total: 12 marks

KEY WORDS riemann sum 黎曼和 • trapezoidal 梯形 • trapezoidal sum 梯形法 • midpoint 中点
• integral 积分

Objective

Build the skills to answer exam questions on **Riemann sums** — estimating the area under a curve with rectangles.

You must be able to:

- compute **left**, **right**, and **midpoint** 中点 Riemann sums from a function or a table
- compute a **trapezoidal** 梯形 estimate
- decide whether an estimate **over-** or **under-estimates** using increasing/decreasing or concavity

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Left and right sums

Estimate $\int_0^6 f \, dx$ with 3 equal subintervals of width $\Delta x = 2$, using the table:

x	0	2	4	6
$f(x)$	1	4	9	16

- **Left** sum uses the left endpoints: $2(1 + 4 + 9) = 28$.
- **Right** sum uses the right endpoints: $2(4 + 9 + 16) = 58$.

■ Midpoint sum

The **midpoint** sum uses the value at the middle of each subinterval. With $f(x) = x^2$ on $[0, 6]$, $n = 3$, $\Delta x = 2$, midpoints 1, 3, 5: $2(f(1) + f(3) + f(5)) = 2(1 + 9 + 25) = 70$.

■ Trapezoidal estimate

A **trapezoidal** sum averages each pair of endpoint heights: $\Delta x \left[\frac{f_0}{2} + f_1 + f_2 + \frac{f_3}{2} \right]$. For the table: $2 \left[\frac{1}{2} + 4 + 9 + \frac{16}{2} \right] = 2(21.5) = 43$. It equals the average of the left and right sums.

■ Over- or under-estimate

For an **increasing** function, a left sum **under**-estimates and a right sum **over**-estimates. For a **concave-up** function, the trapezoidal sum **over**-estimates and the midpoint sum **under**-estimates.

2 Practice

Now apply the methods above.

Use the table for 2.1–2.3:

x	0	3	6	9
$g(x)$	2	5	6	10

2.1 Find the **left** Riemann sum for $\int_0^9 g \, dx$ with the three subintervals shown. [2]

2.2 Find the **right** Riemann sum for the same integral. [2]

2.3 Find the **trapezoidal** estimate. [2]

3 Exam-style questions

3.1 For an increasing function, a **left** Riemann sum gives an [1]

- **A** over-estimate
- **B** under-estimate
- **C** exact value

- **D** value that depends on concavity

3.2 A car's speed $v(t)$ in m/s is recorded:

t (s)	0	4	8	12
$v(t)$	0	12	20	24

(a) Estimate the distance travelled over $0 \leq t \leq 12$ using a **right** Riemann sum with three subintervals. [2]

(b) Is this an over- or under-estimate of the true distance? Give a reason. [1]

3.3 The function f is concave up on $[0, 4]$. A student estimates $\int_0^4 f \, dx$ with a trapezoidal sum. State, with a reason, whether the estimate is too large or too small. [2]

Solutions

2.1 $\Delta x = 3$; left $= 3(2 + 5 + 6) = 39$.

2.2 right $= 3(5 + 6 + 10) = 63$.

2.3 $3\left[\frac{2}{2} + 5 + 6 + \frac{10}{2}\right] = 3(17) = 51$. (Equivalently the average of 39 and 63.)

3.1 B — for an increasing function the left sum under-estimates.

3.2 (a) $4(12 + 20 + 24) = 224$ m. (b) Over-estimate — v is increasing, so right endpoints are the largest values on each subinterval.

3.3 Too large — for a concave-up curve each trapezoid lies **above** the curve, so the trapezoidal sum over-estimates.