AP CALCULUS BC • EXERCISE SHEET

6.13 Evaluating Improper Integrals

Name: _____ Class: _____ Date: _____ **Total: 13 marks**

KEY WORDS diverges 发散 • improper integral 反常积分 • limit 极限 • converges 收敛 • integral 积分

Objective

Build the skills to answer exam questions on **improper integrals** — integrals with an infinite limit or an infinite discontinuity.

You must be able to:

- rewrite an improper integral as a **limit** 极限
- evaluate the limit to decide **convergence** 收敛 or **divergence** 发散
- handle a discontinuity inside the interval

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Infinite upper limit

Replace ∞ with b and take a limit:

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^{b} x^{-2}\,dx = \lim_{b\to\infty} \Big[-\tfrac{1}{x}\Big]_1^{b} = \lim_{b\to\infty} \Big(-\tfrac{1}{b}+1\Big) = 1.$$

The limit is finite, so the integral **converges** to 1.

■ A divergent example

$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \big[\ln x\big]_1^{b} = \lim_{b\to\infty} \ln b = \infty$. The limit is infinite, so it **diverges**.

■ Discontinuity at an endpoint

$\int_0^{1} \frac{1}{\sqrt{x}}\,dx$ is improper at $x = 0$: $\lim_{a\to 0^+} \int_a^1 x^{-1/2}\,dx = \lim_{a\to 0^+} \big[2\sqrt{x}\big]_a^1 = 2 - 0 = 2$. Converges.

■ The rule for 1/x^p on [1, ∞)

$\int_1^{\infty} \frac{1}{x^p}\,dx$ **converges** when $p > 1$ and **diverges** when $p \le 1$ — a handy check.

2 Practice

Now apply the methods above.

2.1 Rewrite $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ as a limit. [1]

2.2 Evaluate $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$. [3]

2.3 State whether $\displaystyle\int_1^\infty \frac{1}{x^{1/2}}\,dx$ converges or diverges (use the p-rule). [1]

3 Exam-style questions

3.1 $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges when [1]

- **A** $p < 1$
- **B** $p = 1$
- **C** $p > 1$
- **D** all p

3.2 Evaluate $\displaystyle\int_0^\infty e^{-x}\,dx$.

(a) Write it as a limit. [1]

(b) Evaluate. [3]

3.3 Determine whether $\displaystyle\int_2^\infty \frac{1}{x}\,dx$ converges or diverges, showing your reasoning with a limit. [3]

Solutions

2.1 $\lim_{b\to\infty}\int_1^b x^{-3}\,dx$.

2.2 $\left[-\frac{1}{2x^2}\right]_1^b = -\frac{1}{2b^2}+\frac{1}{2}$; as $b\to\infty$ this $\to \frac{1}{2}$.

2.3 Diverges — $p=\frac{1}{2}\le 1$.

3.1 **C** — converges for $p>1$.

3.2 (a) $\lim_{b\to\infty}\int_0^b e^{-x}\,dx$. (b) $\left[-e^{-x}\right]_0^b = -e^{-b}+1$; $\to 1$ as $b\to\infty$.

3.3 $\lim_{b\to\infty}\left[\ln x\right]_2^b = \lim_{b\to\infty}(\ln b-\ln 2)$; $\ln b\to\infty$, so it **diverges**.