AP CALCULUS BC • EXERCISE SHEET

10.8 Ratio Test for Convergence

Name: ______________ Class: __________ Date: ___________ **Total: 14 marks**

KEY WORDS ratio test 比值判别法 • converges absolutely 绝对收敛 • converges 收敛 • diverges 发散

Objective

Build the skills to answer exam questions on the **ratio test** for convergence.

You must be able to:

- compute $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$
- conclude: $L < 1$ converges, $L > 1$ diverges, $L = 1$ inconclusive
- apply it to series with **factorials** or n**th powers**

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The ratio test

Compute

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

- $L < 1$: **converges absolutely**;
- $L > 1$: **diverges**;
- $L = 1$: **inconclusive**.

■ A factorial example

$\sum \frac{2^n}{n!}$: $\frac{a_{n+1}}{a_n} = \frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2}{n+1} \to 0 < 1$. **Converges**.

■ An n-th power example

$\sum \frac{n}{2^n}$: $\frac{a_{n+1}}{a_n} = \frac{(n+1)/2^{n+1}}{n/2^n} = \frac{n+1}{2n} \to \frac{1}{2} < 1$. **Converges**.

■ When it is inconclusive

For a p-series, $L = 1$, so the ratio test says nothing — use the p-series rule instead. The ratio test shines with factorials and exponentials.

2 Practice

Now apply the methods above.

2.1 For $\sum \frac{3^n}{n!}$, write $\frac{a_{n+1}}{a_n}$. [2]

2.2 Find $\lim_{n\to\infty} \frac{3}{n+1}$ and state the conclusion. [2]

2.3 For a p-series, the ratio test gives $L = 1$. What does this mean? [1]

3 Exam-style questions

3.1 If the ratio test gives $L = 1$, the series [1]

- **A** converges
- **B** diverges
- **C** the test is inconclusive
- **D** equals 1

3.2 Apply the ratio test to $\sum_{n=1}^{\infty} \frac{n!}{10^n}$.

(a) Find $\frac{a_{n+1}}{a_n}$ and its limit. [3]

(b) State the conclusion. [1]

3.3 Use the ratio test on $\sum \frac{5^n}{n^2}$. [4]

Solutions

2.1 $\dfrac{a_{n+1}}{a_n} = \dfrac{3^{n+1}/(n+1)!}{3^n/n!} = \dfrac{3}{n+1}$.

2.2 $\dfrac{3}{n+1} \to 0 < 1$, so the series **converges**.

2.3 The ratio test is inconclusive; use another test (the p-series rule).

3.1 **C** — inconclusive when $L = 1$.

3.2 (a) $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!/10^{n+1}}{n!/10^n} = \dfrac{n+1}{10} \to \infty$. (b) $L > 1$, so it **diverges**.

3.3 $\dfrac{a_{n+1}}{a_n} = \dfrac{5^{n+1}/(n+1)^2}{5^n/n^2} = 5 \cdot \dfrac{n^2}{(n+1)^2} \to 5$; $L = 5 > 1$, so it **diverges**.