AP CALCULUS BC • EXERCISE SHEET

10.4 Integral Test for Convergence

Name: ______________ Class: __________ Date: ____________ **Total: 14 marks**

KEY WORDS integral test 积分判别法 • converges 收敛 • diverges 发散 • geometric series 几何级数 • harmonic series 调和级数

Objective

Build the skills to answer exam questions on the **integral test** for convergence.

You must be able to:

- check the conditions (positive, decreasing, continuous)
- compare $\sum a_n$ with $\int_1^\infty f(x)\,dx$
- conclude convergence or divergence from the integral

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The integral test

If $a_n = f(n)$ where f is **positive, decreasing, and continuous** for $x \geq 1$, then $\sum a_n$ and $\int_1^\infty f(x)\,dx$ **both converge or both diverge**.

A worked test

$\sum \dfrac{1}{n^2}$: with $f(x) = \frac{1}{x^2}$, $\int_1^\infty x^{-2}\,dx = 1$ (finite). The integral converges, so the **series converges**.

A divergent case

$\sum \dfrac{1}{n}$: $\int_1^\infty \dfrac{1}{x}\,dx = \lim\limits_{b\to\infty} \ln b = \infty$. The integral diverges, so the series diverges.

Check the conditions first

The function must be positive and decreasing (eventually). The integral's **value** is not the series' sum — only its convergence matches.

2 Practice

Now apply the methods above.

2.1 State the three conditions on f for the integral test. [2]

2.2 Evaluate $\int_1^{\infty} \frac{1}{x^2}\, dx$. [2]

2.3 From 2.2, does $\sum \frac{1}{n^2}$ converge? [1]

3 Exam-style questions

3.1 The integral test compares a series with [1]

- **A** a geometric series
- **B** an improper integral of the matching function
- **C** the harmonic series
- **D** a partial sum

3.2 Apply the integral test to $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

(a) Evaluate $\int_1^{\infty} \frac{1}{x^3}\, dx$. [3]

(b) State the conclusion. [1]

3.3 Use the integral test to decide whether $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges. (Use the substitution $u = \ln x$.) [4]

Solutions

2.1 Positive, decreasing, and continuous for $x \geq 1$.

2.2 $\left[-\frac{1}{x}\right]_1^\infty = 0 - (-1) = 1$.

2.3 Yes — the integral converges, so the series converges.

3.1 **B** — with an improper integral of the matching function.

3.2 (a) $\displaystyle\int_1^\infty x^{-3}\,dx = \left[-\frac{1}{2x^2}\right]_1^\infty = 0 + \frac{1}{2} = \frac{1}{2}$. (b) Converges.

3.3 $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx$; $u = \ln x$, $du = \frac{1}{x}\,dx$; $= \int \frac{1}{u}\,du = \ln|u| = \ln(\ln x)$; as $x \to \infty$ this $\to \infty$, so it **diverges**.