

10.15 Representing Functions as Power Series

Name: _____ Class: _____ Date: _____

Total: 14 marks

KEY WORDS converges 收敛 • power series 幂级数 • radius of convergence 收敛半径 • geometric series 几何级数

Objective

Build the skills to answer exam questions on **representing functions as power series** — building new series by calculus.

You must be able to:

- **differentiate** or **integrate** a known power series term by term
- build a new series from the geometric series
- use a series to approximate an integral

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Term-by-term calculus

Within its radius of convergence, a power series can be **differentiated** and **integrated** term by term to get the series of the derivative or antiderivative.

■ Building a new series by integrating

Start from $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Integrating gives

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

■ Building by substitution then integrating

$\frac{1}{1+x^2} = \sum (-1)^n x^{2n}$. Integrating term by term gives the series for $\arctan x$:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

■ Approximating an integral

A function like e^{-x^2} has no elementary antiderivative, but its **series** integrates term by term, letting you approximate $\int_0^1 e^{-x^2} dx$.

2 Practice

Now apply the methods above.

2.1 Differentiate the series $\sum_{n=0}^{\infty} x^n$ term by term. [2]

2.2 Write the Maclaurin series for $\frac{1}{1-x}$ and integrate its first three terms. [2]

2.3 State the first three nonzero terms of the series for $\arctan x$. [1]

3 Exam-style questions

3.1 Within its radius of convergence, a power series may be [1]

- **A** differentiated but not integrated
- **B** integrated but not differentiated
- **C** both differentiated and integrated term by term
- **D** neither

3.2 Start from $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(a) Write the series for e^{-x^2} . [2]

(b) Write the first three terms of $\int_0^1 e^{-x^2} dx$ obtained by integrating that series term by

term.

[3]

3.3 Use the geometric series to write a power series for $\frac{1}{1-2x}$, and state its radius of convergence. [3]

Solutions

$$\mathbf{2.1} \quad \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} nx^{n-1}.$$

$$\mathbf{2.2} \quad \sum x^n = 1 + x + x^2 + \cdots; \text{integrating: } x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots.$$

$$\mathbf{2.3} \quad x - \frac{x^3}{3} + \frac{x^5}{5}.$$

3.1 C — both, term by term, within the radius.

$$\mathbf{3.2} \quad (\text{a}) \quad e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2} - \cdots. \quad (\text{b}) \quad \int_0^1 (1 - x^2 + \frac{x^4}{2} - \cdots) dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \cdots \right]_0^1 = 1 - \frac{1}{3} + \frac{1}{10} - \cdots.$$

$$\mathbf{3.3} \quad \frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n; \text{converges for } |2x| < 1, \text{ i.e. } R = \frac{1}{2}.$$