

10.13 Radius and Interval of Convergence of Power Series

Name: _____ Class: _____ Date: _____

Total: 15 marks

KEY WORDS interval of convergence 收敛区间 • radius of convergence 收敛半径 • converges 收敛
 • ratio test 比值判别法 • diverges 发散

Objective

Build the skills to answer exam questions on the **radius and interval of convergence** of a power series.

You must be able to:

- use the **ratio test** to find the **radius of convergence** 收敛半径 R
- test the **endpoints** separately
- state the full **interval of convergence** 收敛区间

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Radius by the ratio test

A power series $\sum c_n(x-a)^n$ converges for $|x-a| < R$. Find R by applying the ratio test and requiring $L < 1$.

■ A worked radius

$\sum \frac{(x-2)^n}{3^n}$: $\left| \frac{a_{n+1}}{a_n} \right| = \frac{|x-2|}{3} \rightarrow \frac{|x-2|}{3}$. Set < 1 : $|x-2| < 3$, so $R = 3$, center 2: the open interval is $(-1, 5)$.

■ Test the endpoints

The ratio test is inconclusive at $x = -1$ and $x = 5$. At $x = 5$: $\sum 1$ diverges; at $x = -1$: $\sum (-1)^n$ diverges. So the interval is the open $(-1, 5)$.

■ Stating the interval

Combine the open interval with each endpoint's separate result, using $[]$ (included) or $()$ (excluded). Always test **both** ends.

2 Practice

Now apply the methods above.

2.1 For $\sum \frac{x^n}{2^n}$, apply the ratio test to find R . [3]

2.2 State the open interval of convergence for a series centered at 0 with $R = 2$. [1]

2.3 Why must the endpoints be tested separately? [1]

3 Exam-style questions

3.1 The radius of convergence is found using the [1]

- **A** integral test
- **B** ratio test
- **C** n th term test
- **D** comparison test

3.2 Consider $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$.

(a) Find the radius of convergence. [3]

(b) Test the endpoints $x = 0$ and $x = 2$. [2]

(c) State the interval of convergence. [1]

3.3 Find the radius of convergence of $\sum \frac{x^n}{n!}$. [3]

Solutions

2.1 $\left| \frac{x^{n+1}/2^{n+1}}{x^n/2^n} \right| = \frac{|x|}{2}$; set < 1 : $|x| < 2$; $R = 2$.

2.2 $(-2, 2)$.

2.3 The ratio test gives $L = 1$ at the endpoints, so it is inconclusive there.

3.1 B — the ratio test gives the radius.

3.2 (a) $\left| \frac{(x-1)^{n+1}/(n+1)}{(x-1)^n/n} \right| = |x-1| \cdot \frac{n}{n+1} \rightarrow |x-1|$; $< 1 \Rightarrow R = 1$. (b) $x = 2$: $\sum \frac{1}{n}$ diverges; $x = 0$: $\sum \frac{(-1)^n}{n}$ converges. (c) $[0, 2)$.

3.3 $\left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \frac{|x|}{n+1} \rightarrow 0 < 1$ for all x ; $R = \infty$ (converges everywhere).