10.1 Defining Convergent and Divergent Infinite Series

Name: __________ Class: ________ Date: _________ **Total: 11 marks**

KEY WORDS partial sums 部分和 • converges 收敛 • diverges 发散

Objective

Build the skills to answer exam questions on **convergent and divergent series** — the meaning of an infinite sum.

You must be able to:

- define convergence through the limit of **partial sums 部分和**
- compute the first few partial sums
- decide convergence when the partial sums have a known limit

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

Partial sums

The Nth **partial sum** of $\sum a_n$ is $S_N = a_1 + a_2 + \cdots + a_N$. The series **converges** to L if $S_N \to L$; otherwise it **diverges**.

Computing partial sums

For $\sum \left(\frac{1}{2}\right)^n$ starting at $n = 1$: $S_1 = \frac{1}{2}$, $S_2 = \frac{3}{4}$, $S_3 = \frac{7}{8}$. The partial sums approach 1, so the series converges to 1.

A telescoping series

$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$: $S_N = 1 - \frac{1}{N+1}$, which $\to 1$. Converges to 1.

Divergence by growing sums

For $\sum 1 = 1 + 1 + 1 + \cdots$: $S_N = N \to \infty$, so it diverges. If the partial sums do not settle to a finite value, the series diverges.

2 Practice

Now apply the methods above.

2.1 Write the third partial sum S_3 of $\sum_{n=1}^{\infty} \frac{1}{n}$. [1]

2.2 A series has partial sums $S_N = 3 - \frac{1}{N}$. State its sum. [1]

2.3 A series has $S_N = 2N$. State whether it converges or diverges. [1]

3 Exam-style questions

3.1 A series $\sum a_n$ converges if [1]

- **A** $a_n \to 0$
- **B** the partial sums S_N approach a finite limit
- **C** a_n is always positive
- **D** there are infinitely many terms

3.2 For the series $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$:

(a) Write S_3. [2]

(b) Find a formula for S_N and hence the sum. [3]

3.3 A series has partial sums $S_N = \dfrac{N}{N+1}$. State the sum of the series, with a reason.[2]

Solutions

2.1 $S_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$.

2.2 Sum $= 3$ (as $N \to \infty$, $\frac{1}{N} \to 0$).

2.3 Diverges ($S_N \to \infty$).

3.1 **B** — convergence is defined by the partial sums approaching a finite limit.

3.2 (a) $S_3 = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) = 1 - \frac{1}{4} = \frac{3}{4}$. (b) $S_N = 1 - \frac{1}{N+1}$; as $N \to \infty$, sum $= 1$.

3.3 $\dfrac{N}{N+1} \to 1$, so the series converges to 1.