Exercise walk-through

Finding the Area of the Region Bounded by Two Polar Curves ▪ 9.9

eXercise

You must be able to

- find the **intersection angles** of two polar curves $(r_1 = r_2)$
- integrate $\dfrac{1}{2}(r_{\text{outer}}^2 - r_{\text{inner}}^2)$
- decide which curve is outer over each interval

Method — The formula

The area between an outer curve r_1 and an inner curve r_2 (over the same θ-range) is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(r_1^2 - r_2^2\right)\,d\theta.$$

Subtract the inner sector from the outer sector.

Method — Finding intersection angles

Set the curves equal. For $r = 3$ and $r = 2 + 2\cos\theta$:
$3 = 2 + 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$.

Method — Which is outer?

Test a θ between the intersections. Between $-\frac{\pi}{3}$ and $\frac{\pi}{3}$, at $\theta = 0$: $r = 3$ vs $r = 4$, so the cardioid $2 + 2\cos\theta$ is outer there.

Method — Setting up

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left((2 + 2\cos\theta)^2 - 3^2\right) d\theta.$$

Use symmetry to write it as $2 \times \frac{1}{2}\int_0^{\pi/3}(\dots)\, d\theta$ if convenient.

Practice 2.1 ▪ 1 mark

State the area-between-two-polar-curves formula.

Solution 2.1

Method: The formula

Worked steps

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(r_1^2 - r_2^2\right)\, d\theta \quad \boxed{\mathbf{B1}}.$$

Practice 2.2 ▪ 3 marks

Find the intersection angles of $r = 2$ and $r = 4\cos\theta$.

Solution 2.2

Method: Finding intersection angles

Worked steps

$2 = 4\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$ M1 M1 A1.

Practice 2.3 ▪ 1 mark

Between $\theta = 0$ and $\theta = \frac{\pi}{6}$, which is larger: $r = 2$ or $r = 4\cos\theta$?

Solution 2.3

Method: Which is outer?

Worked steps

$r = 4\cos\theta$ is larger (at $\theta = 0$: $4 > 2$) **B1**.

Exam-style 3.1 ▪ 1 mark

The area between an outer curve r_1 and inner r_2 is

A $\dfrac{1}{2}\displaystyle\int (r_1 - r_2)^2 \, d\theta$

B $\dfrac{1}{2}\displaystyle\int (r_1^2 - r_2^2) \, d\theta$

C $\displaystyle\int (r_1^2 - r_2^2) \, d\theta$

D $\dfrac{1}{2}\displaystyle\int (r_1^2 + r_2^2) \, d\theta$

Solution 3.1

Method: The formula

A $\frac{1}{2}\int (r_1 - r_2)^2 \, d\theta$

B $\frac{1}{2}\int (r_1^2 - r_2^2) \, d\theta$ ✓

C $\int (r_1^2 - r_2^2) \, d\theta$

D $\frac{1}{2}\int (r_1^2 + r_2^2) \, d\theta$

Worked steps

B — half the integral of the difference of the squared radii.

Exam-style 3.2 ▪ 2 marks

Two curves are $r = 1$ (inner) and $r = 2\cos\theta$ (outer) where they overlap.
(a) Find the intersection angles.
(b) Set up the area inside $r = 2\cos\theta$ and outside $r = 1$.

Solution 3.2

Worked steps

(a) $1 = 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$ M1 A1. (b) $A = \dfrac{1}{2}\displaystyle\int_{-\pi/3}^{\pi/3} \left((2\cos\theta)^2 - 1^2\right) d\theta$ M1 M1 A1.

Exam-style 3.3 ▪ 2 marks

Explain why you must square the radii **before** subtracting, i.e. why $r_1^2 - r_2^2 \neq (r_1 - r_2)^2$ in the area formula.

Solution 3.3

Method: The formula

Worked steps

Each sector's area is $\frac{1}{2}r^2\,d\theta$, so the region between is $\frac{1}{2}(r_1^2 - r_2^2)\,d\theta$ — a difference of the two **squared** radii; $(r_1 - r_2)^2$ would wrongly square the difference of the radii instead **M1** **A1**.